Fourier Transform of a Uniform Disk

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A uniform disk, that is: a filled circle of uniform intensity, has a Fourier transform that can be expressed analytically.

The Fourier transform of a disk of uniform intensity, where the intensity is equal to unity, is:

$$A(f) = \frac{\sqrt{3r}}{4f} J_1(2\pi f r) \tag{1}$$

where r is the radius of the disk, f is the frequency in cycles per unit length (the same units r is measured in). $J_1(x)$ is a Bessel function of the first kind. The leading coefficient is there to make the integral of $|A(f)|^2$ over all space be equal to the area of the circle times its brightness squared.[1] This is simply a statement of conservation of power between coordinate transforms. Multiplying the circle's brightness by some factor simply requires one multiply $A(f)$ by the same factor. Thus, if the circle is twice as bright, then $A(f)$ is twice as large.

It should be noted that since $A(f)$ is the Fourier amplitude as a function of frequency for a radially symmetric object that the total power, i.e. its integral over all space is given in polar coordinates:

$$TotalPower = \int_0^\infty 2\pi |A(f)|^2 df \tag{2}$$

In interferometry a more useful statement is the visibility of a uniform disk where the visibility is, by convention, equal to unity at zero spatial frequency. This is given by:

$$V = 2\left(\frac{J_1(2\pi f r)}{2\pi f r}\right) \tag{3}$$

where the expression has been left unsimplified to allow easier comparison with other versions of this formula in the literature.

This information puts us in a position to compute at what frequency the visibility reaches its first null given the size of a uniform disk. We begin by noting that

$$J_1(3.83) \simeq 0 \tag{4}$$

Thus, we can solve for the value of spatial frequency, f, where the first null occurs.

$$\begin{aligned} J_1(2\pi f r) &= 0 && (5)\\ 2\pi f r &= 3.83 && (6)\\ f &= \frac{1.22}{2r} && (7) \end{aligned}$$

where the 2 is placed in the denominator to reveal the familiar factor of 1.22 cited so often in relation to an Airy disk (which is simply $|A(f)|^2$). The above equation is true if r and f are in common units. For example, if r is the size of a star in radians, then f would be spatial frequency in units of cycles/radian. In our group we commonly measure stellar sizes is arcsec. and spatial frequency in SFU. In these units the above equation becomes:

$$f = \frac{1.26}{r} \tag{8}$$

[1] Here again, the brightness is equal to unity.