Astro 161: Cosmology

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Nutritional Facts:
Serving size: 1 Semester (16 weeks)
Servings per container: many problems and solutions

Problem # 1 Time Evolution of $\Omega$

a) From the Friedmann equation and the relation $\rho \propto a^{-3(1+w)}$ (for component $i$), derive

$$1 - \Omega(a) = \frac{1 - \Omega_0}{1 - \Omega_0 + \Omega_{0,\Lambda}a^2 + \Omega_{0,m}a^{-1} + \Omega_{0,r}a^{-2}}$$

where $\Omega_0 = \Omega(a = 1) = \Omega(a)$ is the present-day value of the total density parameter $\Omega(a)$.

First we need the Friedmann equation and also the relation between the density parameter and the energy density

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}\pi G \rho - \frac{\kappa c^2}{a^2} \quad \Omega(a) = \frac{\rho(a)}{\rho_c}$$

to find an expression for the Hubble parameter in terms we use the critical density derived for a flat universe (where $\kappa = 0$) which is

$$\rho_c = \frac{3H^2}{8\pi G} \quad \text{therefore} \quad \Omega(a)H^2 = \frac{8}{3}\pi G \rho \quad \text{and} \quad \Omega_0H_0^2 = \frac{8}{3}\pi G \rho_0$$

we can find the curvature parameter by using the Friedmann equation

$$H_0^2 = \frac{8}{3}\pi G \rho_0 - \frac{\kappa c^2}{a^2} \quad \text{therefore} \quad \kappa = \frac{H_0^2}{c^2} (\Omega_0 - 1)$$

we also know

$$\Omega_0 = \sum_i \Omega_{0,i} \quad \text{and} \quad \rho_0 = \sum_i a^{-3(1+w_i)} \quad \text{and} \quad H^2 \Omega = \frac{8}{3}\pi G \rho = H_0^2 \Omega_0 a^{-3(1+w)}$$

using these relationships we can write the Friedmann equation as

$$H^2 = H_0^2 \sum \Omega_{0,1} a^{-3(1+w_i)} - \frac{H_0^2 \sum_i \Omega_{0,i} - 1}{a^2} = H_0^2 \left( \sum_i \Omega_{0,i} a^{-3(1+w_i)} - \frac{\sum_i \Omega_{0,i} - 1}{a^2} \right)$$

(1)
to find the relationship between $H^2$ and $\Omega(a)$ we need to use the Friedmann equation

$$H^2 = H^2 \Omega(a) - \frac{\kappa c^2}{a^2} \Rightarrow H^2 = H^2 \Omega(a) + \frac{H_0^2 (1 - \Omega_0)}{a^2}$$

thus

$$H^2 (1 - \Omega(a)) = \frac{H_0^2 (1 - \Omega_0)}{a^2} \Rightarrow H^2 = \frac{H_0^2 (1 - \Omega_0)}{(1 - \Omega(a))a^2}$$
plugging the above results into equation 1 and factoring a $a^{-2}$ from equation 1 we find

$$\frac{H_0^2 (1 - \Omega_0)}{(1 - \Omega(a)) a^2} = \frac{H_0^2}{a^2} \left( \sum_i \Omega_{0,i} a^{-1 - 3w_i} + 1 - \sum_i \Omega_{0,i} \right)$$

expanding the sum in terms of the three components of the universe we can write this as

$$\frac{(1 - \Omega_0)}{(1 - \Omega(a))} = (\Omega_{0,\Lambda} a^2 + \Omega_{0,m} a^{-1} + \Omega_{0,r} a^{-2} + 1 - \Omega_0)$$

and solving for $(1 - \Omega(a))$ we find

$$1 - \Omega(a) = \frac{1 - \Omega_0}{1 - \Omega_0 + \Omega_{0,\Lambda} a^2 + \Omega_{0,m} a^{-1} + \Omega_{0,r} a^{-2}}$$

which is what we were trying to derive.

b) What is the asymptotic value of $\Omega(a)$ toward the big bang? Does it depend on $\Omega_0$ or the individual $\Omega_{0,i}$?

rewriting the solution to the evolution of $\Omega(a)$ as

$$\Omega(a) = 1 - \frac{1 - \Omega_0}{1 - \Omega_0 + \Omega_{0,\Lambda} a^2 + \Omega_{0,m} a^{-1} + \Omega_{0,r} a^{-2}}$$

and taking the limit as $a \to 0$ we find that $\Omega(a) \to 1$. and this show that this is only dependent on the individual components.

c) Compute $\Omega(a)$ at $a = 10^{-3}$ for four values of $(\Omega_{0,m}, \Omega_{0,\Lambda}):(0.3,0.0),(0.3,0.7),(1.0,0.0), \text{ and } (3.0,0.0)$. (The universe at $a = 10^{-3}$ is matter-dominated to a good approximation, so you can ignore radiation) Plot $\Omega(a)$ for the four cases.

case 1
$\Omega_{0,m} = 0.3$ and $\Omega_{0,\Lambda} = 0.0$ therefore $\Omega_0 = 0.3$ and therefore

$$\Omega(a) = 1 - \frac{1 - 0.3}{1 - 0.3 + 300} = \boxed{0.9967}$$

case 2
$\Omega_{0,m} = 0.3$ and $\Omega_{0,\Lambda} = 0.7$ therefore $\Omega_0 = 1.0$ and therefore

$$\Omega(a) = \boxed{1}$$

case 3
$\Omega_{0,m} = 1.0$ and $\Omega_{0,\Lambda} = 0.0$ therefore $\Omega_0 = 1.0$ and therefore

$$\Omega(a) = \boxed{1}$$

case 4
$\Omega_{0,m} = 3.0$ and $\Omega_{0,\Lambda} = 0.0$ therefore $\Omega_0 = 3.0$ and therefore

$$\Omega(a) = 1 - \frac{1 - 3}{1 - 3 + 3000} = \boxed{1.00067}$$

and the plots are
d) From the parametric solutions to the Friedmann equation, show that $a(t) \propto t^\beta$ at small $t$ for both open and closed models. (Again, consider only matter-dominated era). Find the value of $\beta$ and compare it to the value for the flat model. Is your answer consistent with (c)?

The parametric solutions to the Friedmann for the open model where ($\kappa < 0$) is

$$H_0t = \frac{\Omega_0}{2(1-\Omega_0)^{3/2}}(\sinh(\theta) - \theta)$$ and $$a(\theta) = \frac{\Omega_0}{2(1-\Omega_0)}(\cosh(\theta) - 1)$$

the taylor expansions for the hyporbolic functions are given by

$$\sinh(\theta) \approx \theta + \frac{\theta^3}{6}$$ and $$\cosh(\theta) \approx 1 + \frac{\theta^2}{2}$$

therefore

$$a(\theta) \propto \theta^2$$ and $$t(\theta) \propto \theta^3$$

thus

$$a \propto t^{2/3} \quad \beta = \frac{2}{3}$$

The parametric solutions to the Friedmann for the closed model where ($\kappa > 0$) is

$$H_0t = \frac{\Omega_0}{2(1-\Omega_0)^{3/2}}(\theta - \sin(\theta))$$ and $$a(\theta) = \frac{\Omega_0}{2(1-\Omega_0)}(1 - \cos(\theta))$$

the taylor expansions for the trigonometric functions are given by
\[
\sin(\theta) \approx \theta - \frac{\theta^3}{6} \quad \text{and} \quad \cos(\theta) \approx 1 - \frac{\theta^2}{2}
\]

therefore

\[
a(\theta) \propto \theta^2 \quad \text{and} \quad t(\theta) \propto \theta^3
\]

thus

\[
a \propto t^{2/3} \quad \beta = \frac{2}{3}
\]

and these answers are consistent with (c)

**Problem #2 Fates and Ages of the Universes**

Depending on the average density, a universe can be open, flat, or closed. Consider three universes with \( \Omega_0 = 0.3, 1.0, \) and \( 3.0 \) respectively. Assume \( H_0 = 70 \text{ km s}^{-1} \text{Mpc}^{-1} \) and a matter-dominated era (i.e. ignore radiation and the cosmological constant).

a) Express \( H_0 \) in units of inverse years

\[
H_0 = 70 \frac{\text{km}}{\text{s} \cdot \text{Mpc}} = \frac{70 \times 10^3 \text{m} \cdot \pi \times 10^7 \text{s}}{3.086 \times 10^{22} \text{m} \cdot \text{s} \cdot \text{yr}} = 7.13 \times 10^{-11} \text{yr}^{-1}
\]

b) For the closed universe, calculate the time (in years) when expansion reaches a maximum, and the time when the “Big Crunch” occurs.

For a closed universe (where \( \kappa > 1 \) and \( \Omega_0 > 1 \)) we have two parametric equations given by

\[
a(\theta) = \frac{1}{2 \Omega_0 - 1} \left( \frac{\Omega_0}{2 H_0} \right)^{3/2} (1 - \cos(\theta))
\]

\[
t(\theta) = \frac{\Omega_0}{2 H_0 (\Omega_0 - 1)^{3/2}} (\theta - \sin(\theta))
\]

we know that the expansion reaches a maximum when \( \theta = \pi \) and therefore

\[
t(\pi) = \frac{3 \pi}{4 \sqrt{2 H_0}} \approx 23.4 \text{ Gyr}
\]

and the time it takes for the “big crunch” is just \( 2 \times t(\pi) \) which is just

\[
46.8 \text{ Gyr}
\]

c) For the flat universe compute \( \dot{a} \) in the limit as \( t \rightarrow \infty \).

Using the relationship between the scale factor \( a(t) \) and \( t \), which is given by

\[
a(t) \propto t^{2/3}
\]

and taking a time derivative we find

\[
\dot{a} = \frac{2}{3} t^{-1/3}
\]

if we take the limit as \( t \rightarrow \infty \) than \( \dot{a} \rightarrow 0 \), therefore
\[
\text{\( t \to \infty \) then \( \dot{a} \to 0 \)}
\]

d) For the open universe use the parametric expression given in lecture to compute \( \dot{a} \) in the limit as \( t \to \infty \). Compare your answer to part (c).

for a open universe (where \( \kappa < 1 \) and \( \Omega_0 < 1 \)) we have two parametric equations given by

\[
a(\theta) = \frac{1}{2} \frac{\Omega_0}{1 - \Omega_0} (\cosh(\theta) - 1) \\
t(\theta) = \frac{1}{2H_0} \frac{\Omega_0}{(1 - \Omega_0)^{3/2}} (\sinh(\theta) - \theta)
\]

let

\[
a = \frac{1}{2} \frac{\Omega_0}{1 - \Omega_0} \text{ and } b = \frac{1}{2H_0} \frac{\Omega_0}{(1 - \Omega_0)^{3/2}}
\]

and taking a derivative of \( a(\theta) \) in time we get

\[
\dot{a} = a (\sinh(\theta) \frac{d\theta}{dt})
\]

now we need to know what \( \frac{d\theta}{dt} \) is and we use the time dependent parametric equation

\[
t(\theta) = b (\sinh(\theta) - \theta) \Rightarrow \frac{d\theta}{dt} = \frac{1}{b (\cosh(\theta) - 1)}
\]

therefore

\[
\dot{a} = \frac{a}{b} \left( \frac{\sinh(\theta)}{\cosh(\theta) - 1} \right) = \frac{a}{b}
\]

and as \( t \to \infty \) then \( \dot{a} \to H_0 \sqrt{1 - \Omega_0} \)

thus

\[
\text{\( t \to \infty \) then \( \dot{a} \to H_0 \sqrt{1 - \Omega_0} \)}
\]

e) Using the fact that \( a(t_0) = 1 \) compute \( t_0 \) for all three models. (Hint: for the open and closed cases, fist solve for \( \theta_0 \) - feel free to plug in numbers
the flat universe is the simplest case and \( t_0 \) is defined as

\[
t_0 = \frac{2}{3H_0} = 9.25 \text{ Gyr}
\]

the next case we can look at is the closed universe, where \( \Omega_0 = 3 \) and \( a(\theta_0) = 1 \), thus

\[
a(\theta) = \frac{1}{2} \frac{\Omega_0}{\Omega_0 - 1} (1 - \cos(\theta)) \Rightarrow 1 = \frac{3}{4} (1 - \cos(\theta)) \Rightarrow \theta_0 = \cos^{-1} \left( -\frac{1}{3} \right) = 1.9016 \text{ rad}
\]

plugging this into the time dependent parametric equation yields
\[ t(\theta) = \frac{1}{2H_0} \frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} (\theta - \sin(\theta)) \Rightarrow t_0 = \frac{3}{4\sqrt{2}H_0} (.937) = 6.97 \text{ Gyr} \]

and finally we look at the open universe, where (\(\Omega_0 = 0.3\) and \(a(\theta_0) = 1\)), thus

\[ a(\theta) = \frac{1}{2} \frac{\Omega_0}{2 - \Omega_0} (\cosh(\theta) - 1) \Rightarrow 1 = \frac{3}{14} (\cosh(\theta) - 1) \Rightarrow \theta_0 = \cosh^{-1}\left(\frac{17}{3}\right) = 2.42 \text{ rad} \]

plugging this into the time dependent parametric equation yields

\[ t(\theta) = \frac{1}{2H_0} \frac{\Omega_0}{(1 - \Omega_0)^{3/2}} (\sinh(\theta) - \theta) \Rightarrow t_0 = \frac{0.512}{2H_0} (3.15) = 11.3 \text{ Gyr} \]

**f)** On a single plot, draw three curves for the scale factor \(a\) vs. proper time \(t\) for the three universes. Use linear scales for both axes.

Line up the \(a(t)\) curves so that \(t_0 = 0\) for **all three models** (i.e. we are shifting the time axis for each model so that all three universe have \(a = 1\) at \(t = 0\) instead of \(a = 0\) at \(t = 0\)). Your time axis should go from -20 Gyr to +20 Gyr.

Be sure to label the time when the Big Bang occurs on each curve (this will occur in the past at negative \(t\)), and the times when the closed universe reaches a maximum and undergoes the Big Crunch.
Problem #3 Hubble Diagram I: Distance vs. Redshift

a) Starting with the Robertson-Walker metric, show that for arbitrary $\Omega_{0,m}$ and $\Omega_{0,\lambda}$ (ignoring radiation), the comoving distance $r$ and redshift $z$ are related by

$$r = |k|^{-1/2} \sinh \left\{ \frac{c|k|^{1/2}}{H_0} \int_0^z \frac{dz'}{\sqrt{\Omega_{0,m}(1+z')^3 + \Omega_{0,\lambda} + (1 - \Omega_{0,m} - \Omega_{0,\lambda})(1+z')^2}} \right\}$$

(2)

where $\sin = \sin$ for $k > 0$, $\sin = \sinh$ for $k < 0$, and $\sin$ is absent for $k = 0$.

the Robertson-Walker metric is defined as

$$ds^2 = c^2 dt^2 - a(t)^2 \left[ \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right]$$

for a photon $ds^2 = 0$ and we know that $\theta = 0$ and $\phi = 0$ and $d\Omega^2 = 0$ for radial comoving coordinates. Therefore we know that the the Robertson-Walker metric becomes

$$c dt = a(t) \frac{dr}{\sqrt{1 - kr^2}}$$

we know that

$$\frac{\dot{a}}{a} = H \Rightarrow dt = \frac{da}{aH} \quad \text{and also} \quad a(t) = \frac{1}{1 + z} \quad \text{and} \quad da = -\frac{1}{(1+z)^2} dz$$

rewriting the R-W metric using this information yields

$$\frac{c da}{a(t)^2 H} = \frac{dr}{\sqrt{1 - kr^2}}$$

we know that $H$ can be written as

$$H = H_0 \sqrt{\Omega_0 a^{-3(1+w)} + (1 - \Omega_0)a^{-2}} = H_0 \sqrt{\Omega_{0,m} a^{-3} + \Omega_{0,\lambda} + (1 - \Omega_{0,m} - \Omega_{0,\lambda})a^{-2}}$$

plugging this into the last equation along with the relationship between $a(t)$ and $z$ we find

$$-\frac{c}{H_0 \sqrt{\Omega_{0,m}(1+z)^3 + \Omega_{0,\lambda} + (1 - \Omega_{0,m} - \Omega_{0,\lambda})(1+z)^2}} dz = \frac{dr}{\sqrt{1 - kr^2}}$$

changing the limits of integration for $z$ to $z'$ we change the sign of the integral, therefore we find

$$\int_0^z \frac{c}{H_0 \sqrt{\Omega_{0,m}(1+z')^3 + \Omega_{0,\lambda} + (1 - \Omega_{0,m} - \Omega_{0,\lambda})(1+z')^2}} dz' = \frac{dr}{\sqrt{1 - kr^2}}$$

(3)

this takes care of the left hand side of the equation. Now for the right hand side, the first case we will look at is for $k = 0$ we find
\[
\int_0^r dr = r \quad \text{therefore} \quad r = \int_0^c \frac{c}{H_0} \frac{dz'}{\sqrt{\Omega_{0,m}(1+z')^3 + \Omega_{0,\lambda} + (1 - \Omega_{0,m} - \Omega_{0,\lambda})(1+z')^2}}
\]

now for a closed universe, where \( \Omega_0 > 1 \) and \( k > 0 \) we find

\[
\int_0^r \frac{1}{\sqrt{1-kr^2}}dr
\]

if we let

\[
x = \sqrt{kr} \quad \Rightarrow \quad dr = \frac{dx}{\sqrt{k}} \quad \text{therefore} \quad \frac{1}{\sqrt{k}} \int \frac{1}{\sqrt{1-x^2}}dx = \frac{1}{\sqrt{k}} \sin^{-1}(x) = \frac{1}{\sqrt{k}} \sin^{-1}(\sqrt{kr})
\]

therefore for \( k > 0 \) we find

\[
\int_0^r \frac{1}{\sqrt{1-kr^2}}dr = \frac{1}{\sqrt{k}} \sin^{-1}(\sqrt{kr})
\]

putting this into equation 2 we find

\[
\frac{1}{\sqrt{k}} \sin^{-1}(\sqrt{kr}) = \int_0^c \frac{c}{H_0} \frac{dz'}{\sqrt{\Omega_{0,m}(1+z')^3 + \Omega_{0,\lambda} + (1 - \Omega_{0,m} - \Omega_{0,\lambda})(1+z')^2}}
\]

and so for \( k > 0 \) we find

\[
r = |k|^{-1/2} \sin \left[ \frac{|c|^{1/2}}{H_0} \int_0^c \frac{dz'}{\sqrt{\Omega_{0,m}(1+z')^3 + \Omega_{0,\lambda} + (1 - \Omega_{0,m} - \Omega_{0,\lambda})(1+z')^2}} \right]
\]

now for an open universe, where \( \Omega_0 < 1 \) and \( k < 0 \) we find

\[
\int_0^r \frac{1}{\sqrt{1+kr^2}}dr
\]

if we let

\[
x = \sqrt{-kr} \quad \Rightarrow \quad dr = \frac{dx}{\sqrt{k}} \quad \text{therefore} \quad \frac{1}{\sqrt{k}} \int \frac{1}{\sqrt{1+x^2}}dx = \frac{1}{\sqrt{k}} \sinh^{-1}(x) = \frac{1}{\sqrt{k}} \sinh^{-1}(\sqrt{-kr})
\]

therefore for \( k < 0 \) we find

\[
\int_0^r \frac{1}{\sqrt{1+kr^2}}dr = \frac{1}{\sqrt{k}} \sinh^{-1}(\sqrt{-kr})
\]

putting this into equation 2 we find
\[
\frac{1}{\sqrt{k}} \sinh^{-1}(\sqrt{kr}) = \int_0^z \frac{c}{H_0} \frac{dz'}{\sqrt{\Omega_{0,m}(1+z')^3 + \Omega_{0,\lambda} + (1 - \Omega_{0,m} - \Omega_{0,\lambda})(1+z')^2}}
\]

and so for \( k < 0 \) we find

\[
\begin{align*}
\begin{array}{c}
\left. \int_0^z \frac{c}{H_0} \frac{dz'}{\sqrt{\Omega_{0,m}(1+z')^3 + \Omega_{0,\lambda} + (1 - \Omega_{0,m} - \Omega_{0,\lambda})(1+z')^2}} \right|_{z=0}^{z'}
\end{array}
\end{align*}
\]

therefore

\[
\begin{align*}
\begin{array}{c}
\left. \int_0^z \frac{c}{H_0} \frac{dz'}{\sqrt{\Omega_{0,m}(1+z')^3 + \Omega_{0,\lambda} + (1 - \Omega_{0,m} - \Omega_{0,\lambda})(1+z')^2}} \right|_{z=0}^{z'}
\end{array}
\end{align*}
\]

where \( sinn = \sin \) for \( k > 0 \), \( sinn = \sinh \) for \( k < 0 \), and \( sinn \) is absent for \( k = 0 \)

b) For small redshift, expand the expression above to show that for all three cases of \( k \),

\[
r = \frac{c}{H_0} \left[ z - \frac{1}{2} (1 + q_0) z^2 \right] + O(z^3)
\]

where \( q_0 \) is the deceleration parameter. As the next problem will show, this is the key equation used in recent measurements of \( q_0 \).

starting with equation one

\[
r = |k|^{-1/2} \sinh \left( \frac{c k^{1/2}}{H_0} \int_0^z \frac{dz'}{\sqrt{\Omega_{0,m}(1+z')^3 + \Omega_{0,\lambda} + (1 - \Omega_{0,m} - \Omega_{0,\lambda})(1+z')^2}} \right)
\]

we know that the Taylor expansions are given by

\[
(1+z')^3 = 1 + 3z' \quad \text{and} \quad (1+z')^2 = 1 + 2z'
\]

putting this into equation 1 yields

\[
r = |k|^{-1/2} \sinh \left( \frac{c k^{1/2}}{H_0} \int_0^z \frac{dz'}{\sqrt{\Omega_{0,m} + 3z'\Omega_{0,m} + \Omega_{0,\lambda} + 1 - \Omega_{0,m} - \Omega_{0,\lambda} + 2z' - 2z'\Omega_{0,m} - 2z'\Omega_{0,\lambda}}} \right)
\]

\[
= |k|^{-1/2} \sinh \left( \frac{c k^{1/2}}{H_0} \int_0^z \frac{dz'}{\sqrt{z'(\Omega_{0,m} - 2\Omega_{0,\lambda} + 2) + 1}} \right)
\]
we also know that

\[ 2q_0 = \Omega_{0,m} - 2\Omega_{0,\lambda} \]

so

\[ r = |k|^{-1/2} \sin\left( \frac{c|k|^{1/2}}{H_0} \int_0^z \frac{dz'}{\sqrt{1 + 2z'(q_0 + 1)}} \right) = |k|^{-1/2} \sin\left( \frac{c|k|^{1/2}}{H_0} \int_0^z (1 + 2z'(q_0 + 1))^{-1/2} dz' \right) \]

a Taylor expansion yields

\[ r = |k|^{-1/2} \sin\left( \frac{c|k|^{1/2}}{H_0} \int_0^z [1 - z'(q_0 + 1)] dz' \right) = |k|^{-1/2} \sin\left( \frac{c|k|^{1/2}}{H_0} \left[ z' - \frac{z^2}{2} (q_0 + 1) \right]_0^z \right) \]

and since the small \( z << 1 \) then \( \sin x = x \) and \( \sinh x = x \), therefore

\[ r = \frac{c}{H_0} \left[ z - \frac{z^2}{2} (q_0 + 1) \right] \]

**Problem # 4**

**Hubble Diagram II: Type Ia Supernova**

a) The *luminosity distance* \( d_L \) of a light source at redshift \( z \) is defined as

\[ d_L(z) \equiv \left( \frac{L}{4\pi S} \right)^{1/2} \]  

where \( S \) is the flux received by the observer, and \( L \) is the intrinsic luminosity of the source at \( z \). Explain why \( d_L \) is related to the comoving distance \( r \) by

\[ d_L = (1 + z) r \]

we know that the luminosity of a source is defined as

\[ L = A_p(t) S \]

where

\[ A_p(t) = 4\pi r^2 \quad \text{if} \quad k = 0 \quad A_p(t) < 4\pi r^2 \quad \text{if} \quad k > 0 \quad A_p(t) > 4\pi r^2 \quad \text{if} \quad k < 0 \]

If a photon starts with energy \( E_e = hc/\lambda_e \) when the scale factor is \( a(t_e) \), by the time we observe it, the wavelength will have grown to

\[ \lambda_0 = \frac{1}{a(t_e)} \lambda_e = (1 + z) \lambda_e \]
and the energy will have fallen by

\[ E_0 = \frac{E_e}{1 + z} \]

and also \( dt_0 = dt_e(1 + z) \) and the luminosity is given by

\[ L = \frac{dE}{dt} = \frac{dE_0}{dt_0} = \frac{dE_e}{dt_e(1 + z)^2} = \frac{L}{(1 + z)^2} \]

therefore, the flux \( S \) can be written in terms of the luminosity as

\[ S = \frac{L}{4\pi r^2(1 + z)^2} \]

thus

\[ dL = r(1 + z) \]

b) Astronomers often use logarithmic scale, the distance modulus \( \mu \), to measure distances. The distance modulus is related to the luminosity distance by

\[ \mu \equiv 5 \log_{10} \frac{d_L}{10 \text{parsec}} \]  

(7)

Using equations (1), (4), and (5), make a plot of \( \mu \) vs. \( z \) for three cosmological models: \((\Omega_{0,m}, \Omega_{0,\lambda}) = (0.29, 0.71), (0.29, 0.0), (1.0, 0.0)\). (Use \( h = 0.73 \) from WMAP for the hubble parameter.) The course website lists \( \mu \) and \( z \) for 16 supernovae compiled from Tables 1 and 5 of Riess et al (2004). Add these data points to your plot. Compare your plot to their Figure 4 and comment (e.g. are they similar? If not, why not?)

for the case where \( \Omega_{0,\lambda} = 0 \) we can use Mattigs formula

\[ r(z) = \frac{c}{H_0} \frac{2}{\Omega_{0,m}^{2/3}} \frac{1}{1 + z} \left[ \Omega_{0,m}z + (\Omega_{0,m} - 2)(\sqrt{1 + \Omega_{0,m} - 1}) \right] \]

for the case where \( \Omega_{0,\lambda} \neq 0 \) and \( \Omega_0 = 1 \)

we use

\[ r(z) = \int_0^z \frac{c}{H_0} \frac{dz'}{\sqrt{\Omega_{0,m}(1 + z')^3 + \Omega_{0,\lambda}(1 - \Omega_{0,m} - \Omega_{0,\lambda})(1 + z')^2}} \]

and we can solve this using a numerical technique. The combined the plot for b and c is
Figure 1: A plot of distance modulus and redshift $z$. The overplotted points are from the HST supernova data.

This data seems to show that we live in a universe where $\Omega_{0,m} = 0.29$ and $\Omega_{0,\lambda} = 0.71$.

**Problem #5**

**Weird Behaviour of the Angular Diameter**

a) The *angular diameter distance* $d_A$ is defined to be $d_A = D/\theta$, where $D$ is the physical size of an object and $\theta$ is its angular diameter (i.e. the angle subtended in the sky). Explain why $d_A$ is related to the comoving radial distance $r$ by $d_A = r/(1+z)$.

the angular diameter distance is defined as

$$d_A \equiv \frac{l}{\delta \theta}$$

and the R-W metric can be written as

$$ds = a(t)c_S(r)\delta \theta$$

where $S(k) = \begin{cases} R_0 \sin(r/R_0) & (k > 1) \\ r & (k = 0) \\ R_0 \sinh(r/R_0) & (k < 1) \end{cases}$
for a standard yardstick whose length is \( l \) is known, we can set \( ds = l \) and thus we find

\[
l = a(t_c) S_k(r) \delta \theta
\]

thus the angular diameter distance \( d_A \) is

\[
d_A = \frac{S_k(r)}{1+z} \quad \text{where} \quad S_k(r) = \frac{d_L}{1+z}
\]

and using equation 5 we find

\[
d_A = \frac{d_L}{(1+z)^2} = \frac{r}{1+z} \quad \text{for} \quad k = 0
\]

b) For an object of physical size \( D \) at redshift \( z \), write down a general expression for its angular diameter as a function of \( D, z, \Omega_{0,m} \), and \( H_0 \) (assume \( \Omega_{0,\lambda} = 0 \) for simplicity). Rewrite the formula so that the angular diameter is in units of \( h \) arcsec (Where \( H_0 = 100h \) km/s/Mpc) and \( D \) in units of kpc.

we know that

\[
d_A = \frac{D}{\theta} = \frac{r}{1+z} \quad \text{thus} \quad \theta = \frac{D(1+z)}{r}
\]

c) The physical size of the luminosity part of a typical spiral galaxy is about 30 kpc. On log-log scales, plot the angular diameter of such a galaxy versus redshift \((0.01 < z < 10)\) for three values of \( \Omega_{0,m} \): \((0.3, 1, 3)\). Comment on any interesting features in your curves. (Again assume \( \Omega_{0,\lambda} = 0 \))

for the special case where \( \Omega_{0,\lambda} = 0 \) we use Mattig's formula to make the plots.

\[
r(z) = \frac{c}{H_0 \Omega_{0,m}^2} \frac{1}{1+z} \left[ \Omega_{0,m} z + (\Omega_{0,m} - 2)(\sqrt{1 + \Omega_{0,m}} - 1) \right]
\]

therefore, the angular diameter is given by

\[
\theta = \frac{H_0 \Omega_{0,m}^2 (1+z)^2 D}{2c} \left[ \Omega_{0,m} z + (\Omega_{0,m} - 2)(\sqrt{1 + \Omega_{0,m}} - 1) \right]^{-1}
\]
Figure 2: A plot of distance modulus and redshift $z$. The overplotted points are from the HST supernova data.

The interesting things we can see from this curve is the fact that as the redshift gets bigger the angular diameter also increases. This seems very interesting when looking at morphologies of high redshift galaxies.

Problem # 6 Age of the Universe

In class, we derived expressions for the age of the universe, $t_0$, in terms of the Hubble and the density parameters.

we know that

$$t_0 = \frac{1}{H_0} \int_0^\infty \frac{1}{1+z} \frac{dz}{\sqrt{\Omega_{0,m}(1+z)^3 + \Omega_{0,\lambda} + (1 - \Omega_{0,m} - \Omega_{0,\lambda})(1+z)^2}}$$

where

$$E(z) \equiv \sqrt{\Omega_{0,m}(1+z)^3 + (1 - \Omega_{0,m} - \Omega_{0,\lambda})(1+z)^2 + \Omega_{0,\lambda}}$$
(a) For $\Omega_\lambda = 0$ models, plot $t_0$ in units of $h^{-1}$ Gyr versus the matter density parameter $\Omega_{0,m}$, for $0 \leq \Omega_{0,m} \leq 2.5$. What is the general trend of $t_0$ as the matter density increases? For $\Omega_{0,\lambda} = 0$ we find $t_0$ to be

$$t_0 = \frac{1}{H_0} \int_0^\infty \frac{1}{1+z} \frac{dz}{E(z)}$$

where

$$E(z) = \sqrt{\Omega_{0,m}(1+z)^3 + (1-\Omega_{0,m})(1+z)^2}$$

The general trend observed for $t_0$ as $\Omega_{0,m}$ increases is observed to decrease this can be seen from the plot of $t_0$ vs $\Omega_{0,m}$ provided on the next page.

(b) On the same plot, add a curve for $t_0$ vs. $\Omega_{0,m}$ for flat models (i.e. $\Omega_{0,m} + \Omega_{0,\lambda} = 1$). For the case where $\Omega_{0,m} + \Omega_{0,\lambda} = 1$ we find that $t_0$ is given by

$$t_0 = \frac{1}{H_0} \int_0^\infty \frac{1}{1+z} \frac{dz}{E(z)}$$

where

$$E(z) = \sqrt{\Omega_{0,m}(1+z)^3 + (1-\Omega_{0,m})}$$

(c) Consider the flat models in (b). Plot $H_0$ in units of km s$^{-1}$Mpc$^{-1}$, versus $\Omega_{0,m}$ (same range as (b)), for 3 values of $t_0$: 11.5, 13.7, 18 Gyr. Current observations find $H_0 \approx 70$ km s$^{-1}$Mpc$^{-1}$, and the oldest objects in the universe is at least 11.5 Gyr old. Given these, what is the constraint on $\Omega_{0,m}$ implied by your curves? For some years, $\Omega_{0,m} = 1$ was the favored value. What would be the constraint on $H_0$ implied by $t_0 > 11.5$ Gyr if $\Omega_{0,m} = 1$? How is it compared with the observed $H_0$?

To plot $H_0$ we can just re-write the Equation 2 as

$$H_0 = \frac{1}{t_0} \int_0^\infty \frac{1}{1+z} \frac{dz}{E(z)}$$

for $t_0 = 11.5, 13, 18$ Gyr we use

$$H_{11.5} = 84.7 \frac{\text{km}}{\text{s Mpc}} \int_0^\infty \frac{1}{1+z} \frac{dz}{E(z)}$$

$$H_{13.0} = 74.92 \frac{\text{km}}{\text{s Mpc}} \int_0^\infty \frac{1}{1+z} \frac{dz}{E(z)}$$

$$H_{18.0} = 54.11 \frac{\text{km}}{\text{s Mpc}} \int_0^\infty \frac{1}{1+z} \frac{dz}{E(z)}$$

and the plots are presented in the following figures.
Current observations find $H_0 \approx 70 \text{ km s}^{-1}\text{Mpc}^{-1}$, and the oldest objects in the universe is at least 11.5 Gyr old. Given these, what is the constraint on $\Omega_{0,m}$ implied by your curves?

From our plot oh $H_0$ vs. $\Omega_{0,m}$ we can see that the constraint implied by our curves on $\Omega_{0,m}$ is

\[ \Omega_{0,m} \leq 0.5 \]

For some years, $\Omega_{0,m} = 1$ was the favored value. What would be the constraint on $H_0$ implied by $t_0 > 11.5$ Gyr if $\Omega_{0,m} = 1$? How is it compared with the observed $H_0$?

From our plot of $H_0$ vs. $\Omega_{0,m}$ we can see that the constraint implied by our curves on $H_0$ if $\Omega_{0,m}$ is 1 is that

\[ H_0 \leq 60 \text{ Km s}^{-1}\text{Mpc}^{-1} \]

and this is approximately

14% less than the observed value for $H_0$

**Problem # 7**

**Most Distant Galaxies**

The most distant objects known are a galaxy at redshift $z = 6.96$ which has been confirmed and also a pseudo galaxy at $z = 10$ which has not been confirmed. Compute the ages of the universe when the light that we recieve today was emitted from these two galaxies for: (1) flat cold dark matter (CDM) model with $\Omega_m = 1$; (2) open CDM model $\Omega_{0,m} = 0.27$, $\Omega_{0,\lambda} = 0$; and (3) flat $\Lambda$CDM model with $\Omega_{0,m} = 0.27$ and $\Omega_{0,\lambda} = 0.73$.

we can define

\[ H_0 = \frac{h}{9.77 \text{ Gyr}} \]

Using QROMB in IDL we can numerically integrate these functions to find
when \( \Omega_m = 1 \) \( \Omega_\lambda = \Omega_r = 0 \) than we can find \( t_0 \) by using

\[
t_{6.96} = \frac{1}{H_0} \int_{6.96}^\infty \frac{1}{1+z} \frac{dz}{E(z)} = \frac{0.03}{H_0} = 0.293 \text{ Gyr h}^{-1}
\]

\[
t_{10} = \frac{1}{H_0} \int_{10}^\infty (1+z)^{-5/2} \frac{dz}{E(z)} = \frac{0.018}{H_0} = 0.176 \text{ Gyr h}^{-1}
\]

when \( \Omega_{0,m} = 0.27 \), \( \Omega_\lambda = \Omega_r = 0 \) than we use Equation 1

\[
t_z = \frac{1}{H_0} \int_z^\infty \frac{dz}{1+z E(z)}
\]

therefore

\[
t_{6.96} = \frac{1}{H_0} \int_{6.96}^\infty \frac{1}{(1+z)^2 E(z)} \frac{dz}{H_0} = \frac{0.0522}{H_0} = 0.510 \text{ Gyr h}^{-1}
\]

\[
t_{10} = \frac{1}{H_0} \int_{10}^\infty \frac{1}{(1+z)^2 E(z)} \frac{dz}{H_0} = \frac{0.0329}{H_0} = 0.321 \text{ Gyr h}^{-1}
\]

when \( \Omega_{0,m} + \Omega_{0,\lambda} = 1 \) than we can use Equation 2

\[
t_z = \frac{1}{H_0} \int_z^\infty \frac{dz}{1+z E(z)}
\]

\[
t_{6.96} = \frac{1}{H_0} \int_{6.96}^\infty \frac{1}{1+z E(z)} \frac{dz}{H_0} = \frac{0.0569}{H_0} = 0.556 \text{ Gyr h}^{-1}
\]

\[
t_{10} = \frac{1}{H_0} \int_{10}^\infty \frac{1}{1+z E(z)} \frac{dz}{H_0} = \frac{0.0350}{H_0} = 0.342 \text{ Gyr h}^{-1}
\]

Problem # 8

Dark Energy Model

One of the more recent speculations in cosmology is that the universe may contain a quantum field, called “quintessence,” which has a positive energy density and a negative value of the equation-of-state parameter \( w \). Assume, for the purposes of this problem, that the universe is spatially flat, and contains nothing but matter \( (w = 0) \), and quintessence with \( w = -\frac{1}{2} \). The current density parameter of matter is \( \Omega_{0,m} \leq 1 \), and the current density parameter of quintessence is \( \Omega_{0,Q} = 1 - \Omega_{0,m} \). At what scale factor \( a_{m,Q} \) will the energy density of quintessence and matter be equal? Solve the Friedmann equation to find \( a(t) \) for this universe. What is \( a(t) \) in the limit \( a \ll a_{m,Q} \)? What is \( a(t) \) in the limit \( a \gg a_{m,Q} \)? What is the current age of this universe, expressed in terms of \( H_0 \) and \( \Omega_{0,m} \)?

if we know

\[
\Omega_{0,Q} = 1 - \Omega_{0,m} \quad w = -\frac{1}{2} \quad \Omega_{0,m} < 1
\]

The Friedmann equation for this universe can be written as
\[
\frac{H^2}{H_0} = \frac{\Omega_{0,m}}{a^3} + \frac{\Omega_{0,Q}}{a^{3/2}} \quad H = H_0 \sqrt{\frac{\Omega_{0,m}}{a^3} + \frac{\Omega_{0,Q}}{a^{3/2}}} \quad H = \frac{da}{adt}
\]

therefore

\[
H_0 dt = \frac{da}{a} \left( \frac{\Omega_{0,m}}{a^3} + \frac{\Omega_{0,Q}}{a^{3/2}} \right)^{-1/2} = \frac{da}{a} \left( \frac{\Omega_{0,m}}{a^3} + \frac{1 - \Omega_{0,m}}{a^{3/2}} \right)^{-1/2}
\]

to find \(a_{m,Q}\) we do

\[
\frac{\Omega_{0,m}}{a^3} = \frac{\Omega_{0,Q}}{a^{3/2}}
\]

thus

\[
a_{m,Q} = \left( \frac{\Omega_{0,m}}{\Omega_{0,Q}} \right)^{2/3} \Rightarrow \frac{\Omega_{0,m}}{\Omega_{0,Q}} = \frac{a_{m,Q}^{3/2}}{a_{m,Q}}
\]

now we need to integrate the Friedmann equation to find \(a(t)\)

\[
H_0 t = \frac{da}{a \sqrt{\frac{\Omega_{0,m}}{a^3} + \frac{\Omega_{0,Q}}{a^{3/2}}}}
\]

then we can do

\[
H_0 t = \int_0^a \frac{da}{\sqrt{\Omega_{0,m} a^{-1} + \Omega_{0,Q} a^{1/2}}} = \int_0^a \frac{\sqrt{a} da}{\sqrt{\Omega_{0,m} a + \Omega_{0,Q} a^{3/2}}}
\]

if we let

\[
u = \Omega_{0,Q} a^{3/2} \quad \sqrt{a} da = \frac{2}{3 \Omega_{0,Q}} du
\]

therefore

\[
\frac{2}{3 \Omega_{0,Q}} \int_0^\left(\frac{\Omega_{0,Q} a \frac{3}{2}}{2}\right) \frac{du}{\sqrt{\Omega_{0,m} + \Omega_{0,Q} a^{3/2}}} \Rightarrow \frac{4}{3 \Omega_{0,Q}} \left[ \sqrt{\Omega_{0,m} + \Omega_{0,Q} a^{3/2}} - \sqrt{\Omega_{0,m}} \right]
\]

and we find

\[
H_0 t = \frac{4}{3 \Omega_{0,Q}} \left[ \sqrt{\Omega_{0,m} + \Omega_{0,Q} a^{3/2}} - \sqrt{\Omega_{0,m}} \right]
\]

\[
\left[\frac{3}{4} H_0 t (1 - \Omega_{0,m}) + \sqrt{\Omega_{0,m}} \right]^2 = \Omega_{0,m} + (1 - \Omega_{0,m}) a^{3/2}
\]

therefore \(a(t)\) is

\[
a(t) = \left[ \left( \frac{3 H_0 t (1 - \Omega_{0,m}) + \sqrt{\Omega_{0,m}}}{4} \right) - \Omega_{0,m} \right] (1 - \Omega_{0,m})^{-1} \right]^{2/3} \quad (10)
\]

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to take the limits we must use a different form of this equation expressed in terms of \( a \) and \( a_{m,Q} \), we can start by using
the expression for \( a_{m,Q} \) in terms of \( \Omega_{0,m} \) and \( \Omega_{0,Q} \)
\[
H_0t = \frac{1}{\sqrt{\Omega_{0,Q}}} \int_0^a \frac{\sqrt{a}da}{\sqrt{\Omega_{0,m} + a_{m,Q}^3}}
\]

letting
\[
u = a_{m,Q}^{3/2} + a^{3/2} \quad \sqrt{a}da = \frac{2}{3}du
\]
thus
\[
H_0t = \frac{2}{3\sqrt{1-\Omega_{0,m}}} \int u^{-1/2}du
\]
\[
H_0t = \frac{4}{3\sqrt{1-\Omega_{0,m}}} \left[ \left( 1 + \left( \frac{a}{a_{m,Q}} \right)^{3/2} \right)^{1/2} - 1 \right]
\]
(11)
to find the limit when \( a \ll a_{mQ} \) we must use a taylor approximation, and we find
\[
H_0t = \frac{2}{3\sqrt{1-\Omega_{0,m}}} a_{m,Q}^{3/4}
\]
and after making the appropriate substitution we find
\[
\left[ \frac{3H_0t}{2} \right]^{2/3}
\]
cubing
\[
a(t) = \left[ \frac{3H_0t \sqrt{\Omega_{0,m}}}{2} \right]^{2/3}
\]
thus \( a \propto t^{2/3} \)
to find the limit as \( a \gg a_{m,Q} \) we can ignore the 1’s in Equation 4 and we find
\[
H_0t = \frac{4}{3\sqrt{1-\Omega_{0,m}}} a_{m,Q}^{3/4}
\]
therefore
\[
\left[ \frac{3H_0t \sqrt{1-\Omega_{0,m}}}{4} \right]^{4/3}
\]
cubing
\[
a(t) = \left[ \frac{3H_0t \sqrt{1-\Omega_{0,m}}}{4} \right]^{4/3}
\]
thus \( a \propto t^{4/3} \)
and last but not least, we can find the current age of this universe by using
\[
H_0t = \frac{4}{3(1-\Omega_{0,m})} \left[ \sqrt{\Omega_{0,m} + (1-\Omega_{0,m})a_{m,Q}^3} - \sqrt{\Omega_{0,m}} \right]
\]
if we set \( a = 1 \) for the value of the scale factor today we find that the current age of this universe is
\[
t_0 = \frac{4}{3H_0} \frac{1-\sqrt{\Omega_{0,m}}}{1-\Omega_{0,m}}
\]
Problem #9 $\pi$ is not a constant

In the Schwarzschild metric $r$ is a **comoving** coordinate, not a real physical distance. Rather, integrals over $ds$ constitute physical distances. In the following we will take slices of spacetime at a constant time $(dt = 0)$.

(a). First, compute the physical circumference, $C$, at a given coordinate distance $R$ from the center of a black hole of mass $M$ at $\theta = \frac{\pi}{2}$.

the Schwarzschild metric is given as

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2 \quad (12)$$

In this problem we know that

$$dt = 0 \quad dr = 0 \quad d\theta = 0 \quad \theta = \frac{\pi}{2}$$

thus the Schwarzschild metric simplyfies to

$$ds^2 = r^2 \sin^2\theta d\phi^2 = r^2d\phi^2$$

therefore to find the circumference we can integrate this function from $0 \leq \phi \leq 2\pi$

$$C = \int_0^{2\pi} Rd\phi = 2\pi R$$

(b). Second, compute the physical distance $R_{\text{phys}}$ from the center of the black hole out to the coordinate distance $R$ (assume $R > R_s$ and take the absolute value of $g_{rr}$).

[Hint: The following facts may be helpfull:

$$\int_0^1 \sqrt{\frac{1}{1-\xi}}d\xi = \frac{\pi}{2}, \quad \int_1^\alpha \sqrt{\frac{1}{1-\xi}}d\xi = \ln(\sqrt{\alpha-1} + \sqrt{\alpha}) + \sqrt{\alpha-1}\sqrt{\alpha} \quad (13)$$

where $\alpha > 1$ is constant.]

for this problem we now have

$$dt = 0 \quad d\theta = 0 \quad d\phi = 0$$

thus the Schwarzschild metric simplyfies to

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right)^{-1}dr^2 \quad \Rightarrow \quad ds = \left(1 - \frac{2GM}{rc^2}\right)^{-1/2}dr$$

this can also be written as

$$ds = \int_0^R \sqrt{\frac{1}{1-\frac{2GM}{rc^2}}}dr$$

if we multiply both top and bottom by $rc^2$ and divide top and bottom by $2GM$ we get an expression of the form
\[ ds = \int_0^R \sqrt{\frac{r c^2}{2GM}} - 1 \, dr \quad \text{let } \xi = \frac{r c^2}{2GM} \Rightarrow \frac{2GM}{c^2} d\xi = dr \]

this can be made to look like Equation 2 by multiplying the denominator by -1 and taking the absolute value of this function

\[ ds = \frac{2GM}{c^2} \int_0^\alpha \sqrt{\frac{\xi}{1-\xi}} d\xi = 2GM \left[ \int_0^1 \sqrt{\frac{\xi}{1-\xi}} d\xi + \int_0^\alpha \sqrt{\frac{\xi}{1-\xi}} d\xi \right] \]

thus

\[ R_{\text{phys}} = \frac{2GM}{c^2} \left[ \frac{\pi}{2} + \ln(\sqrt{\alpha - 1} + \sqrt{\alpha}) + \sqrt{\alpha - 1} \sqrt{\alpha} \right] \quad \alpha = \frac{Rc^2}{2GM} \]

(c). Now use your answers to part (a) and part (b) to compute \( \Pi \) where

\[ C = 2\Pi R_{\text{phys}} \]

using our solution in part (a)

\[ C = 2\pi R \]

we find

\[ 2\pi R = 2\Pi R_{\text{phys}} = 2\Pi \frac{2GM}{c^2} \left[ \frac{\pi}{2} + \ln(\sqrt{\frac{Rc^2}{2GM}} - 1 + \sqrt{\frac{Rc^2}{2GM}} + \sqrt{\frac{Rc^2}{2GM}} - 1) \right] \]

and solving for \( \Pi \) we find

\[ \Pi = \frac{\pi Rc^2}{2GM} \left[ \frac{\pi}{2} + \ln(\sqrt{\frac{Rc^2}{2GM}} - 1 + \sqrt{\frac{Rc^2}{2GM}} + \sqrt{\frac{Rc^2}{2GM}} - 1) \right]^{-1} \]

which can also be written as

\[ \Pi = \pi \xi \left[ \frac{\pi}{2} + \ln(\sqrt{\xi-1} + \sqrt{\xi}) + \sqrt{\xi-1} \sqrt{\xi} \right]^{-1} \]

(d). Plot \( \Pi \) as a function of \( \xi \equiv R/r_s \) for \( \xi \in [1, 10^3] \) (use log axes for the x axis). What happens to \( \Pi \) as \( \xi \to \infty \)?

the plot is given in figure 1.

and we can see that \( \Pi \) approaches the true value of \( \pi \) that is measured in a flat space.
Figure 3: Plot of $\Pi$ versus $\xi$.

$\xi \equiv R/r_s$
Problem #10 [Time to fall into a black hole]

We haven’t spent too much time discussing black holes in great detail. But take it as a given that a particle initially at rest at infinity follows a trajectory that always obeys

\[
\left(1 - \frac{2GM}{rc^2}\right) \frac{dt}{d\tau} = 1
\]

this is a constant of the motion, and is actually closely related to the total energy of the particle.

(a). Rewrite the Schwarzschild metric in the context of a particle falling in from infinity along a direct radial line. Your metric should only have a \(d\tau\) term and a \(dr\) term.

we know

\[d\phi = 0 \quad d\theta = 0\]

thus the Schwarzschild metric simplyfies to

\[ds^2 = -(1 - \frac{2GM}{rc^2}) c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2\]

but we know that

\[dt = d\tau \left(1 - \frac{2GM}{rc^2}\right)^{-1}\]

\[\Rightarrow dt^2 = d\tau^2 \left(1 - \frac{2GM}{rc^2}\right)^{-2}\]

plugging this into equation 3 we find

\[ds^2 = \left(1 - \frac{2GM}{rc^2}\right)^{-1} \left[-c^2 d\tau^2 + dr^2\right]\]

(b). Recall that \(d\tau\) measures the proper time, i.e. the time measured by the particle as it falls into the black hole. Integrate the Schwarzschild metric to compute how long the particle must wait, after it crosses the event horizon, to reach the singularity at the center.

if

\[d\tau^2 = \frac{-ds^2}{c^2}\]

\[\Rightarrow ds^2 = -c^2 d\tau^2\]

we can use this and substitute this into the solution for part (a) to get

\[-c^2 d\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right)^{-1} c^2 d\tau^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2\]

a bit of algebra yields
\[ c^2 d\tau^2 \left[ \left( 1 - \frac{2GM}{rc^2} \right)^{-1} - 1 \right] = \left( 1 - \frac{2GM}{rc^2} \right)^{-1} dr^2 \]

this simplifies further to yield

\[ c^2 d\tau^2 \left[ \frac{rc^2 - rc^2 + 2GM}{rc^2 - 2GM} \right] = \left[ \frac{rc^2}{rc^2 - 2GM} \right] dr^2 \]

\[ c^2 d\tau^2[2GM] = rc^2 dr^2 \]

\[ d\tau = \pm \sqrt{\frac{r}{2GM}} dr \]

we can solve this integral by using

\[ \int_0^\tau d\tau = \int_{r_s}^0 -\sqrt{\frac{r}{2GM}} dr \quad r_s = \frac{2GM}{c^2} \]

\[ \tau = \frac{2}{3} \left( \frac{r_s^3}{2GM} \right)^{1/2} = \left[ \frac{4GM}{3c^3} \right] \]

(c). How long would it take to reach the center of a black hole with mass \( M = 1M_{\odot} \)? How about \( M = 10^6M_{\odot} \)(the balck hole at the center of the milky way)? How about a truly supermassive black hole at the center of a distant quasar with \( M = 10^{10}M_{\odot} \)?

If

\[ G = 6.673 \times 10^{-11} \text{m}^3\text{kg}^{-1}\text{s}^{-2} \quad c = 2.998 \times 10^8 \text{ms}^{-1} \quad M_{\odot} = 1.989 \times 10^{30} \text{kg} \]

for \( M = 1M_{\odot} \) we find

\[ \tau = \frac{4GM}{3c^3} = \frac{4[6.673 \times 10^{-11} \text{m}^3\text{kg}^{-1}\text{s}^{-2}][1.989 \times 10^{30} \text{kg}]}{3[2.998 \times 10^8 \text{ms}^{-1}]^3} = 6.57 \mu s \]

for \( M = 10^6M_{\odot} \) we find

\[ \tau = \frac{4GM}{3c^3} = \frac{4[6.673 \times 10^{-11} \text{m}^3\text{kg}^{-1}\text{s}^{-2}] \cdot 10^6[1.989 \times 10^{30} \text{kg}]}{3[2.998 \times 10^8 \text{ms}^{-1}]^3} = 6.57 \text{ s} \]

for \( M = 10^{10}M_{\odot} \) we find

\[ \tau = \frac{4GM}{3c^3} = \frac{4[6.673 \times 10^{-11} \text{m}^3\text{kg}^{-1}\text{s}^{-2}] \cdot 10^{10}[1.989 \times 10^{30} \text{kg}]}{3[2.998 \times 10^8 \text{ms}^{-1}]^3} = 66.57 \text{ ks} \]
Problem #11 [A Painful Derivation]

Inexplicably, astrophysicists have tended to fixate on the black hole’s ability to shred everything that falls into it. The technical term for this is sphagettification (I’m serious). Simply put, gravitational tidal forces in the vicinity of a black hole are so strong that any object of finite size will experience serious tension as that part of the body which is closest to the black hole is pulled in with more force than that part of the body which is further out.

(a). We’ll analyze the tidal forces on a particle falling in from rest at infinity. Using your metric from part (a) in question 2, derive an expression for $\frac{d\tau}{d\tau}$. This is the velocity of the infalling observer.

Using the result from part (b) we know that

$$d\tau = \sqrt{\frac{r}{2GM}} dr$$

therefore

$$\frac{dr}{d\tau} = \sqrt{\frac{2GM}{r}}$$

(b). Compute the acceleration of the infalling observer

$$a = \frac{d^2r}{d\tau^2}$$

we must use the chain rule to solve this problem, if

$$\frac{dr}{d\tau} = f(r) \quad \text{then} \quad \frac{d^2r}{d\tau^2} = \frac{df(r)}{dr} f(r)$$

and so we find

$$\frac{d^2r}{d\tau^2} = \frac{1}{2} \sqrt{\frac{r}{2GM}} \left( -\frac{2GM}{r^2} \right) \frac{dr}{d\tau} = \frac{1}{2} \sqrt{\frac{r}{2GM}} \left( -\frac{2GM}{r^2} \right) \sqrt{\frac{2GM}{r}} = -\frac{GM}{r^2}$$

therefore

$$a = -\frac{GM}{r^2} = -\frac{c^2 r_s}{2 \tau^2}$$

this is just the acceleration of an object due to a mass $M$ given by Newtonian mechanics.

(c). Now, compute the tidal acceleration $\frac{da}{dr}$. What would the relative acceleration across your body if you were at the event horizon of a solar mass black hole with your feet pointing towards the center? How about for the $10^6 M_{\odot}$ black hole at the center of the milky way? How about for the truly supermassive black hole at the center of a distant quasar with $M = 10^{10} M_{\odot}$? Express your relative accelerations in units of $g = 9.81 \, m/s^2$.

To find the tidal acceleration we just take the derivative of the expression for $a$ given in part (b)
\[
\frac{da}{dr} = \frac{2GM}{r^3}
\]

if we assume that the individual is 2 meters tall and that his feet has length 0 at the edge of the event horizon, we can express the relative acceleration as

\[
da = \int_{r_s}^{r+2} \frac{2GM}{r^3} dr = GM \left[ \frac{1}{r_s^2} - \frac{1}{(r_s + 2)^2} \right]
\]

for \( M = 1M_{\text{sun}} \) we find

\[
r_s = \frac{2GM}{c^2} = \frac{2[6.673 \times 10^{-11} \text{m}^3 \text{kg}^{-1} \text{s}^{-2}][1.989 \times 10^{30} \text{kg}]}{[2.998 \times 10^8 \text{ms}^{-1}]^2} = 2966.77 \text{ m}
\]

\[
da = [6.673 \times 10^{-11} \text{m}^3 \text{kg}^{-1} \text{s}^{-2}][1.989 \times 10^{30} \text{kg}] \left[ \frac{1}{(2966.77)^2} - \frac{1}{(2966.77 + 2)^2} \right]
= 2.04 \times 10^9 \text{g}
\]

for \( M = 10^6 M_{\text{sun}} \) we find

\[
r_s = \frac{2GM}{c^2} = \frac{2[6.673 \times 10^{-11} \text{m}^3 \text{kg}^{-1} \text{s}^{-2}] \cdot 10^6 \cdot [1.989 \times 10^{30} \text{kg}]}{[2.998 \times 10^8 \text{ms}^{-1}]^2} = 2.97 \times 10^9 \text{ m}
\]

\[
da = [6.673 \times 10^{-11} \text{m}^3 \text{kg}^{-1} \text{s}^{-2}] \cdot 10^6 \cdot [1.989 \times 10^{30} \text{kg}] \left[ \frac{1}{(2.97 \times 10^9)^2} - \frac{1}{(2.97 \times 10^9 + 2)^2} \right]
= 2.07 \times 10^{-3} \text{g}
\]

for \( M = 10^{10} M_{\text{sun}} \) we find

\[
r_s = \frac{2GM}{c^2} = \frac{2[6.673 \times 10^{-11} \text{m}^3 \text{kg}^{-1} \text{s}^{-2}] \cdot 10^{10} \cdot [1.989 \times 10^{30} \text{kg}]}{[2.998 \times 10^8 \text{ms}^{-1}]^2} = 2.97 \times 10^{13} \text{ m}
\]

\[
da = [6.673 \times 10^{-11} \text{m}^3 \text{kg}^{-1} \text{s}^{-2}] \cdot 10^{10} \cdot [1.989 \times 10^{30} \text{kg}] \left[ \frac{1}{(2.97 \times 10^{13})^2} - \frac{1}{(2.97 \times 10^{13} + 2)^2} \right]
= 2.08 \times 10^{-11} \text{g}
\]

\( \text{(d). In the case of the } 10^6 M_{\text{sun}} \text{ black hole, for how long would your journey into the center, after crossing the event horizon, be relatively pain free? How about for the } 10^{10} M_{\text{sun}} \text{ black hole?} \)

We will use the result acquired in problem 2, which is
\[ \tau = -\sqrt{\frac{r}{2GM}} dr = -\frac{1}{\sqrt{2GM}} \int_{r_s}^{r_\alpha} r^{1/2} dr \]

where \( r_\alpha \) is the radius at which a human begins feeling pain and is defined as

\[ r_\alpha = \left( \frac{2GM}{g} \Delta r \right)^{1/3} \text{ where } \Delta r = 2m \]

so after integrating the above expression we find

\[ \tau = \frac{2}{3\sqrt{2GM}} \left[ r_s^{3/2} - r_\alpha^{3/2} \right] = \frac{2}{3\sqrt{2GM}} \left[ r_s^{3/2} - r_\alpha^{3/2} \right] \]

for \( M = 10^6M_{\odot} \) we find

\[ r_s = 2.97 \times 10^9 \text{ m} \quad r_\alpha = 3.78 \times 10^8 \text{ m} \]

thus

\[ \tau = \frac{2}{3\sqrt{2[6.673 \times 10^{-11}\text{m}^3\text{kg}^{-1}\text{s}^{-2}] \cdot 10^6 \cdot [1.989 \times 10^{30}\text{kg}]} \left[ (2.97 \times 10^9 \text{ m})^{3/2} - (3.78 \times 10^8 \text{ m})^{3/2} \right] \]

\[ = 6.37 \text{ s} \]

for \( M = 10^{10}M_{\odot} \) we find

\[ r_s = 2.97 \times 10^{13} \text{ m} \quad r_\alpha = 8.14 \times 10^9 \text{ m} \]

thus

\[ \tau = \frac{2}{3\sqrt{2[6.673 \times 10^{-11}\text{m}^3\text{kg}^{-1}\text{s}^{-2}] \cdot 10^{10} \cdot [1.989 \times 10^{30}\text{kg}]} \left[ (2.97 \times 10^{13} \text{ m})^{3/2} - (8.14 \times 10^9 \text{ m})^{3/2} \right] \]

\[ = 66.38 \text{ ks} \]

**Problem #12 [\( \Omega \) in Photon Background Radiation]**

The present temperature of the cosmic background photons is known to exquisite precision

\[ T_0 = 2.725 \pm 0.0001(1\sigma) \text{ Kelvin (Mather et al 1999).} \]

(a). Calculate the present number density of photons (in \( \text{cm}^{-3} \)) in this radiation. (Don’t worry about the error bars.)

we know that the number density for bosons is given by

\[ n_b = \frac{g_b}{\pi^2} \zeta(3) \left( \frac{kT}{\hbar c} \right)^3 = \frac{g_b \cdot \zeta(3)}{\pi^2} \frac{k^3}{\hbar^3 c^3} T^3 \]
where
\[ g_b = 2 \quad \zeta(3) = 1.202 \quad T = 2.725 \text{ Kelvin} \]

therefore we find the number density of these photons to be
\[ n_b = \frac{2.2404}{\pi^2} \frac{k^3}{h^3 c^3} \cdot (2.725)^3 = \boxed{411 \text{ cm}^{-3}} \]

(b). Calculate \( \Omega_{0,\gamma} \), the present ratio of the mass-energy density in photons to the critical density. (Express your answer in terms of the Hubble parameter \( h \).) Is the universe today radiation or matter dominated?

the mass-energy density for bosons is given by
\[ \epsilon_\gamma = \alpha T^4 \quad \alpha = \frac{\pi^2 g_b k^4}{30 \hbar^3 c^3} = \frac{\pi^2}{15 \hbar^3 c^3} T^4 \]
\[ \rho_c = \frac{3H^2 c^2}{8\pi G} \]

The ratio \( \Omega_{0,\gamma} \) is given by
\[ \Omega_{0,\gamma} = \frac{\epsilon_\gamma}{\rho_c} = \frac{\pi^2}{15 \hbar^3 c^3} T^4 \cdot \frac{8\pi G}{3H^2 c^2} = \frac{8\pi^3 k^4 GT^4}{45H^2 c^5} = \frac{8}{45} \frac{\pi^3 k^4 G(2.725 \text{ K})^4}{H^2 c^5 \hbar^3} \]
where
\[ H_0 = 3.241 \times 10^{-18} \text{ s}^{-1} \]

thus we find
\[ \Omega_{0,\gamma} = \boxed{2.47 \times 10^{-5} \text{ h}^{-2}} \]

From this results we can see that the universe is **matter dominated** because \( \Omega_{0,m} \gg \Omega_{0,\gamma} \).

**Problem #13 [Ionizing Photons]**

Suppose the photon temperature \( T \) of a background distribution such that \( kT \ll Q \), where \( Q = 13.6 \text{ eV} \) is the ionization energy of hydrogen.

(a). What fraction \( f \) of the blackbody photons are energetic enough to ionize hydrogen?

The distribution function for bosons is given as
\[ n_\gamma = \frac{4\pi g_b}{(2\pi \hbar)^3} \int_0^\infty \frac{p^2}{e^{cp/\hbar} - 1} dp \]

if we let
\[ E = cp \quad dp = \frac{dE}{c} \]
if \( kT \ll Q \)

we find

\[
n_\gamma = \frac{4\pi g_b}{(2\pi\hbar)^3} \int_Q^{\infty} E^2 e^{-E/kT} dE
\]

using integration by parts we find

\[
n_\gamma = [0 - \left( -e^{-Q/kT} kT(2kT^2 + 2kQT + Q^2) \right) = \frac{4\pi g_b}{(2\pi\hbar c)^3} \left[ e^{-Q/kT} kT(2kT^2 + 2kQT + Q^2) \right]
\]

and the total number density is given by

\[
n_{\gamma T} g_b \frac{\zeta(3)}{(\pi^2)} \left( \frac{kT}{\hbar c} \right)^3
\]

therefore the fraction \( f \) is given by

\[
f = \frac{n_\gamma}{n_{\gamma T}} = \frac{e^{-Q/kT} [2kT^2 + 2kQT + Q^2]}{2k^2 T^2 \zeta(3) (\pi^2)}
\]

(b). As we will discuss later, the cosmic microwave background photons came from the last scattering surface when the universe was at \( T \approx 3700 \) K. What is the numerical value of \( f \) at \( T \approx 3700 \) K? What is \( f \) today?

for \( T = 3700 \) Kelvin \( kT = .3188 \) eV

and \( f \) is found by using

\[
f(3700) = \frac{e^{-Q/kT} [2kT^2 + 2kQT + Q^2]}{2k^2 T^2 \zeta(3)} = 2.35 \times 10^{-16}
\]

and for the value today is given by

\( T = 2.725 \) Kelvin \( kT = 2.35 \times 10^{-4} \) eV

\[
f(2.725) = \frac{e^{-Q/kT} [2kT^2 + 2kQT + Q^2]}{2k^2 T^2 \zeta(3)} \approx 0
\]
Problem #14 [Gamow’s Prediction of the CMB]

A fascinating bit of cosmological history is that of George Gamow’s prediction of the Cosmic Microwave Background in 1948. (Unfortunately, his prediction was premature; by the time the CMB was actually discovered in the 1960’s, his prediction had fallen into obscurity.) Let’s see if you can reproduce Gamow’s line of argument. Gamow knew that nucleosynthesis must have taken place at a temperature \( T_{\text{nuc}} \approx 10^9 \) K, and that the age of the universe is currently \( t_0 \approx 10 \) Gyr.

Assume that the universe is flat and contains only radiation. With these assumptions, what was the energy density \( \varepsilon \) at the time of nucleosynthesis? What was the Hubble parameter \( H \) at the time of nucleosynthesis? What was the time \( t_{\text{nuc}} \) at which nucleosynthesis took place? What is the current temperature \( T_0 \) of the radiation filling the universe today? If the universe switched from being radiation-dominated to being matter-dominated at a redshift \( z_{r,m} > 0 \), will this increase \( T_0 \) for fixed values of \( T_{\text{nuc}} \) and \( t_0 \)? Explain your answer.

The energy density is given by

\[
\varepsilon = \frac{\pi^2 g_b k^4 T^4}{30 \hbar^3 c^3} = \frac{\pi^2 k^4 T^4}{15 \hbar^3 c^3}
\]

for \( T_{\text{nuc}} = 10^9 \) K we find

\[
\varepsilon_{\text{NC}} = \frac{\pi^2 k^4 T^4}{15 \hbar^3 c^3} = \frac{\pi^2 [1.381 \times 10^{-23} \text{JK}^{-1}]^4 [10^9 \text{K}]^4}{15 [1.055 \times 10^{-34} \text{Js}]^3 [3 \times 10^8 \text{ms}^{-1}]^3} = 7.54 \times 10^{20} \text{Jm}^{-3}
\]

The Hubble parameter \( H \) at the time of nucleosynthesis is given by

\[
H_{\text{NC}}^2 = \frac{8\pi G}{3c^2} \varepsilon_{\text{NC}}
\]

therefore \( H \) is given as

\[
H_{\text{NC}} = \sqrt{\frac{8\pi G}{3c^2} \varepsilon_{\text{NC}}} = 0.0022 \text{ s}^{-1}
\]

\( t_{\text{nuc}} \) can be found by using

\[
t_{\text{nuc}} = \frac{1}{2H_{\text{NC}}} = 230.91 \text{ s}
\]

for \( T \) we use

\[
T(t) = \left( \frac{45}{32\pi^2} \right) T_p \left( \frac{t_p}{t} \right)^{1/2} \approx 0.61 T_p \left( \frac{t_p}{t} \right)^{1/2}
\]

where \( T_p = 1.4 \times 10^{32} \) K is the Planck temperature and \( t_p \sim 5 \times 10^{-44} \) s is the Planck time and \( t_0 = 10 \) Gyr = 3.15 \times 10^{17} \) s

thus
\[ T_0 = 0.61 \, T_p \left( \frac{t_p}{t_0} \right)^{1/2} = 34.024 \, \text{K} \]

To explain why the temperature of the matter dominated universe is less than the temperature of the radiation dominated universe we can use

\[ T \propto \frac{1}{a(t)} \]

and since we know

\[ a(t)_r \propto t^{1/2} \quad a(t)_m \propto t^{2/3} \]

so for any time \( t \) we know that

\[ a(t)_r < a(t)_m \]

plugging this into the temperature equation we find that

\[ T_{rD} > T_{mD} \]

thus we just showed that the temperature for radiation dominated universe is greater than the temperature of a matter dominated universe. It follows from the simple analogy that the bigger the volume the lower the temperature. Since \( a(t) \) for a matter dominated universe is greater than \( a(t) \) for radiation dominated we know that the volume of a matter dominated universe is greater than for a radiation dominated universe, and thus the temperature is lower.

**Problem #15 [Ω in the Neutrino Background Radiation]**

In problem set 5, you calculated \( \Omega \) in the cosmic background radiation. Let’s do it for the cosmic neutrinos.

**(a)**. From the photon temperature \( T = 2.725 \) Kelvin, calculate the present-day temperature of the neutrino background

we know that

\[ T_\nu = 0.714 T_\gamma = 1.945 \, \text{Kelvin} \]

**(b)**. Calculate the present-day number density of the electron neutrinos \( \nu_e \) and their antiparticle \( \bar{\nu}_e \) in \( \text{cm}^{-3} \).

for relativistic particles, the number density of neutrinos is defined as

\[ n_f = \frac{3}{4} \frac{g_f}{g_b} n_b \quad n_b = \frac{g_b}{\pi^2} \zeta(3) \left( \frac{kT}{\hbar c} \right)^3 \]
and since we are only considering one species we find $g_{\nu} = 2 = g_f$ so we find

$$n_{\nu_e} = \frac{3}{2\pi^2} \zeta(3) \left( \frac{kT_{\nu_e}}{\hbar c} \right)^3 = 113.27 \text{ cm}^{-3}$$

(c). If the masses of all three species of neutrinos (i.e. $\nu_e, \nu_\mu, \nu_\tau$) are much less than 1 MeV, explain how the number densities of $\nu_\mu$ and $\nu_\tau$ compare to your answers for $\nu_e$ in part (b).

the number densities would all be the same because they are still relativistic and $g_f$ would be the same for all of them.

(d). For what (approximate) range of masses (in eV) do neutrinos behave like relativistic particles today? If neutrinos are relativistic today, then the total background radiation comes from photons, 3 species of left-handed neutrinos, and their corresponding anti-neutrinos. Calculate $\Omega_{0,\nu}$, the present-day density parameter for this radiation.

we know that relativistic particles need to have

$$kT > m c^2 \quad kT = 1.676 \times 10^{-4} \text{ eV}$$

therefore, the range of masses are given by

$$m c^2 < 1.676 \times 10^{-4} \text{ eV}$$

and $\Omega_{0,r}$ is given by

$$\Omega_{0,r} = \frac{\varepsilon_r}{\rho_c}$$

and

$$\varepsilon_r = \varepsilon_b + \varepsilon_\nu = \alpha_b T_{rad}^4 + 2N_\nu \times \left( \frac{7}{16} \right) T_\nu \quad T_\nu = T_{rad} \left( \frac{4}{11} \right)^{1/3}$$

and

$$\alpha_b = \frac{\pi^2 g_b k^4}{30 \hbar^3 c^3}$$

and we find the energy density to be

$$\varepsilon_r = \alpha_b T_{rad}^4 + 2N_\nu \times \left( \frac{7}{16} \right) \alpha_b \left( \frac{4}{11} \right)^{4/3} T_{rad}^4 = 1.68 \alpha_b T_{rad}^4 = 6.98 \times 10^{-14} \text{ J m}^{-3}$$

where $N_\nu$ is the number of neutrino species, for the critical density we find

$$\rho_c = \frac{3H^2 c^2}{8\pi G} = 1.69 \times 10^{-9} \text{ Jh}^2 \text{ m}^{-3}$$

therefore

$$\Omega_{0,r} = \frac{\varepsilon_r}{\rho_c} = 4.13 \times 10^{-5} \text{ h}^{-2}$$
(e). If the neutrinos are non-relativistic today (and have masses \( \ll 1 \text{ MeV} \)), then show that \( \Omega_{0,\nu} \), the contribution of the neutrinos to the present-day density parameter, is related to the neutrino masses by

\[
\Omega_{0,\nu} h^2 = \sum_i \frac{m_i}{\alpha(\text{eV})}
\]

where the index \( i = \nu_e, \nu_\mu, \) and \( \nu_\tau \), and calculate \( \alpha \) (in units of eV)

we know that the mass density of neutrinos can be found using

\[
\rho_\nu = \sum_i m_i n_\nu
\]

and

\[
\rho_c = \frac{3H^2}{8\pi G}
\]

so we can show that

\[
\Omega_{0,\nu} = \frac{\rho_\nu}{\rho_c} = \frac{\sum_i m_i n_\nu 8\pi G}{3H^2} \Rightarrow \Omega_{0,\nu} h^2 = \frac{\sum_i m_i}{\alpha(\text{eV})}
\]

therefore \( \alpha \) is

\[
\alpha = \frac{3H^2}{8\pi G n_\nu} = 93.14 \text{ eV c}^{-2}
\]

**Problem #16 [Equality time]**

(a). Radiation-Matter Equality:

From your answer to problem 1(d) above, calculate the equality redshift, \( z_{eq} \), at which the energy density in matter equals that in radiation. (Leave \( \Omega_{0,m} \) and \( h \) as free variables in your answer.) Approximately how old was the universe at \( z_{eq} \), for a model of \( \Omega_{0,m} = 0.27, \Omega_{0,\Lambda} = 0.73 \), and \( h = 0.7 \)

we know that

\[
\rho_r = \rho_{r,0} a^{-4} \quad \rho_m = \rho_{m,0} a^{-3}
\]

so

\[
\Omega_{0,r} a^{-4} = \Omega_{0,m} a^{-3} \Rightarrow a = \frac{\Omega_{0,r}}{\Omega_{0,m}}
\]

so \( z_{eq} \) is given by

\[
z_{eq} = \Omega_{0,m} (2.41 \times 10^4 h^2) - 1 \approx 3187
\]

if \( h = 0.732 \) we would get \( z_{eq} \approx 3500 \) which is the quoted result in most cosmology textbooks. The time of this universe at \( z_{eq} = 3187 \) and using QROMB in IDL we find (Note: QROMB gives a rough approximation due to the fact that we only integrated from \( z_{eq} \) to \( 1.0 \times 10^5 \).)

\[
t_0 = \frac{1}{H_0} \int_z^{\infty} \frac{dz'}{1+z'_{eq}} \frac{1}{\sqrt{0.27(1+z')^3+0.73}} \approx 99,300 \text{ yr}
\]
(b). Matter-Λ Equality:
For the same model (i.e. $\Omega_{0,m} = 0.27$, $\Omega_{0,\Lambda} = 0.73$), calculate the redshift at which the energy density in matter equals that in the cosmological constant.

we know that

$$\rho_\Lambda = \rho_{\Lambda,0} \quad \rho_m = \rho_{m,0}a^{-3}$$

so

$$\Omega_{0,\Lambda} = \Omega_{0,m}a^{-3} \Rightarrow a = \left(\frac{\Omega_{0,m}}{\Omega_{0,\Lambda}}\right)^{1/3}$$

so $z_{eq}$ is given by

$$z_{eq} = \left(\frac{\Omega_{0,\Lambda}}{\Omega_{0,m}}\right)^{1/3} - 1 \approx 0.39$$

and for the time of this universe, using QROMB

$$t_0 = \frac{1}{H_0} \int_{z}^{\infty} \frac{dz'}{1+z' E(z)} \approx 9.79 \text{ Gyr}$$

(c). Make a simple sketch showing how the energy density in radiation, matter, and $\Lambda$ evolves with the expansion factor $a$ (use log-log scale), and indicate the crossing times.

---

where $z_{eq}(1)$ is the calculated value, (Assuming that $h$ is 0.70), and $t_{eq}(1)$ is calculated from $z_{eq}(1)$. $z_{eq}(2)$ and $t_{eq}(2)$ are the values most generally accepted by cosmologist, given a cosmology derived from recent observations.
Problem #17 [Stopping Cosmic Neutrinos]

The typical energy of a neutrion in the Cosmic Neutrino Background, as pointed out in Chapter 5, is $E_\nu \sim kT_\nu \sim 5 \times 10^{-4}$ eV. What is the approximate interaction cross-section $\sigma_w$ for one of these cosmic neutrinos? Suppose you had a large lump of $^{56}$Fe (with density $\rho = 7900$ kg m$^{-3}$). What is the number density of protons, neutrons, and electrons within the lump of iron? How far, on average, would a cosmic neutrino travel through the iron before interacting with a proton, neutron, or electron? (Assume that the cross-section is simply $\sigma_w$, regardless of the type of particle the neutrino interacts with.)

we know that the interaction cross-section is defined as

\[
\sigma_w = 10^{-47} m^2 \left( \frac{kT}{1 \text{ MeV}} \right)^2 = 2.5 \times 10^{-66} m^2
\]

the number densities is given by

\[
n = \frac{\rho}{m_{Fe}} \quad m_{Fe} = 56 \cdot m_p = 56(1.6726 \times 10^{-27} \text{kg atom}^{-1}) = 9.3677 \times 10^{-26} \text{kg atom}^{-1}
\]

\[
n_{56Fe} = \frac{7900 \text{ kg}}{9.3677 \times 10^{-26} \text{kg}} \text{atom m}^{-3} = 8.43 \times 10^{28} \text{atom m}^{-3}
\]

\[
n_p = n_e = 26 \cdot n_{56Fe} = 2.19 \times 10^{30} \text{atom m}^{-3}
\]

\[
n_n = 30 \cdot n_{56Fe} = 2.53 \times 10^{30} \text{atom m}^{-3}
\]

and to find how far a cosmic neutrino will travel before interacting with a particle can be found using

\[
d = \frac{1}{\sigma_w n_t} \quad n_t = n_p + n_n + n_e = 6.91 \times 10^{30} \text{atom m}^{-3}
\]

therefore

\[
d = \frac{m^3}{2.5 \times 10^{-66} m^2 \cdot 6.91 \times 10^{30}} = 5.79 \times 10^{34} \text{m} = 6.15 \times 10^{18} \text{lyr}
\]

Problem #18 [$\Omega$ in Baryons]

In problem set 5 and 6, you calculated $\Omega_0$ in the cosmic photons and neutrinos. Let’s look at baryons.
show that $\Omega_{0,B}$, the contribution of baryons to the present-day density parameter, is related to the baryon-to-proton ratio $\eta \equiv n_B/n_\gamma$ by

$$\Omega_{0,B} h^2 = \beta \eta$$

Calculate $\beta$ using your (correct) result in Problem 1 of Problem Set 5 (if you got it wrong, make sure to redo it.). Current measurements of light elemental abundances indicate $4.7 \times 10^{-10} \leq \eta \leq 6.5 \times 10^{-10}$ (95% CL). What is the implied $\Omega_{0,B}$? Is it large enough to make the universe flat?

we know that

$$\Omega_{B,0} = \frac{\epsilon_{B,0}}{\epsilon_c} = \frac{\rho_{B,0}}{\rho_c} = \frac{n_B m_B}{\rho_c}$$

and

$$\epsilon_{c,0} = \frac{3c^2H_0^2}{8\pi G} \quad \rho_c = \frac{3(3.241 \times 10^{-18}h s^{-1})^2}{8\pi G} = 1.878 \times 10^{-29} \text{g/cm}^{-3}$$

and we also know

$$\Omega_{\gamma,0} = \frac{\epsilon_{\gamma,0}}{\epsilon_c} = \frac{\rho_{\gamma,0}}{\rho_c} \quad \eta = \frac{n_B}{n_\gamma} \quad n_B = \frac{m_p + m_n}{2}$$

where

$$n_\gamma = \frac{2}{\pi^2} \zeta(3) \left( \frac{k_B T_\gamma}{\hbar c} \right)^3 = 414 \text{ photons/cm}^{-3}$$

so we find

$$\Omega_{B,0} h^2 = \frac{n_B}{n_\gamma} \frac{8\pi G}{3(3.241 \times 10^{-18}h s^{-1})^2} (414 \text{ photons/cm}^{-3}) m_p = \beta \eta$$

thus

$$\beta = \frac{8\pi G}{3(3.241 \times 10^{-18}h s^{-1})^2} (414 \text{ photons/cm}^{-3}) m_p = 6.652 \times 10^7$$

**Problem #19 [Weighing the Universe]**

One of the primary goals of every galaxy survey is to measure the distribution of the luminosities of galaxies, which has been found to be well parameterized by the Schecter luminosity function

$$\phi(L) dL = \phi_* \left( \frac{L}{L_*} \right)^\alpha e^{-L/L_*} d \left( \frac{L}{L_*} \right)$$

where $\phi(L) dL$ is the number density of galaxies with luminosity between $L$ and $L + dL$, and $\phi_*$, $L_*$, and $\alpha$ are parameters to be determined from galaxy surveys.

(a). Show that the luminosity density of galaxies, $j$, is related to $\phi_*, L_*$ and $\alpha$ by a simple algebraic expression. Derive the expression. (Hint: If your answer contains an integral, you haven’t tried hard enough.)

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first we must begin with

\[ j = \int_0^\infty \phi_* \left( \frac{L}{L_*} \right)^\alpha e^{-L/L_*} d \left( \frac{L}{L_*} \right) = L_* \int_0^\infty \phi_* \left( \frac{L}{L_*} \right)^{\alpha+1} e^{-L/L_*} d \left( \frac{L}{L_*} \right) \]

if we let

\[ x = \frac{L}{L_*} \quad dx = d \left( \frac{L}{L_*} \right) \]

and so

\[ j = L_* \int_0^\infty \phi_* x^{\alpha+1} e^{-x} dx \]

this looks like a gamma function,

\[ \int_0^\infty x^{\alpha+1} e^{-x} dx \]

therefore

\[ j = \phi_* L_\alpha \Gamma(\alpha+2) = \phi_* L_\alpha (\alpha+1)! \]

(b). The Sloan Digital Sky Survey gives \( \phi_* = 0.0149 \pm 0.0004 h^3 \text{Mpc}^{-3} \), \( \alpha = -1.05 \pm 0.01 \), and \( L_* = 1.2 \times 10^{10} h^{-2} L_{\odot} \). Calculate \( j \). (Don’t worry about error bars.). What fraction of this luminosity density is contributed by galaxies fainter than \( L_* \)?

given these values we can just plug them into our previous equation, and we find

\[ j = 1.84 \times 10^8 \frac{h L_{\odot}}{\text{Mpc}} \]

now we need to find what \( j \) is for the low luminosity galaxies

\[ j_{L < L_*} = \int_0^{L_*} L_\alpha \phi_* x^{\alpha+1} e^{-x} dx \]

the numerical solution of the gamma function yields

\[ j_{L < L_*} = L_* \phi_* (0.674306) \]

thus

\[ \frac{j_{L < L_*}}{j} = \frac{0.674}{1.031} = 0.6537 \]

(c). The classical technique for quantifying the amount of dark vs. luminous matter is to determine the mass-to-light ratio, \( M/L \) (in units of \( M_{\odot}/L_{\odot} \)). Assuming the luminosity density from part (b) is representative for the universe (the Sloan survey covers a large volume, so this should be an ok assumption), calculate the value of \( M/L \) (in units of \( M_{\odot}/L_{\odot} \)) required to close the universe (i.e. \( \Omega_m = 1 \)). 10s in galaxies and a few 100s in galaxy clusters. Is there enough dark matter to make \( \Omega_m = 1 \)?
The value of $M/L$ is typically measured to be a few 10s in individual galaxies and a few 100s in galaxy clusters. Is there enough dark matter to make $\Omega_m = 1$?

First we know that

$$\Omega_m = 1 = \frac{\rho_m}{\rho_c} = \frac{M}{V \rho_c} \quad \rho_c = \frac{M}{V}$$

Thus

$$\frac{M}{L} = \frac{M}{jV} = \frac{\rho_c}{j} = \frac{3(3.241 \times 10^{-18} h_0^{-1})^2}{8\pi G} \frac{1}{\phi_\alpha L_\alpha (\alpha + 1)!} \approx 1500 \frac{M_{\text{sun}}}{L_{\text{sun}}}$$

So we see that there is not enough dark matter to make $\Omega_m = 1$. This is a lot larger than what we observe in nature.

**Problem #20 [Why Are Massive Neutrinos Considered Hot?]**

(a). Show that the average momentum of cosmic background neutrinos at temperature $T_\nu$ is given by

$$\langle p \rangle \propto k T_\nu / c$$

and compute the constant of proportionality. (Assume $m_\nu < 1 \text{ MeV}$ so they were relativistic when they coupled.)

We know that

$$\langle p \rangle c = \langle E \rangle \quad \langle E \rangle = \frac{\varepsilon_\nu}{n_\nu}$$

And we know that

$$\varepsilon_\nu = \frac{7}{8} g \frac{\pi^2}{30} \left( \frac{k_b T}{\hbar c} \right)^4 \frac{\hbar c}{n_\nu} = 3 \frac{g}{48} \frac{\zeta(3)}{\pi^2} \frac{(k_b T)}{\hbar c}$$

Thus

$$\langle p \rangle c = \frac{\langle E \rangle}{n_\nu c} = \frac{7}{6} \frac{\pi^4}{30 \zeta(3)} k_b T = \frac{k_b T}{c} \gamma$$

Where $\gamma$ is the constant of proportionality which is given by

$$\gamma = \frac{7 \frac{\pi^4}{6} 30 \zeta(3)}{3.15}$$

(b). Using the non-relativistic expression $p = m_\nu v$, show that the average neutrino speed can be written as

$$\langle v \rangle = \beta \left( \frac{1 \text{ eV}}{m_\nu} \right) \left( \frac{T_\nu}{1.945} \right)$$
and compute the constant $\beta$ in km/s.

if we know that

$$p = m_\nu v_\nu$$

then we know that

$$\langle v \rangle = \frac{\langle p \rangle}{m_\nu} = \frac{k_B T_\nu}{m_\nu c} = \beta \left( \frac{1 \text{ eV}}{m_\nu} \right) \left( \frac{T_\nu}{1.945} \right)$$

thus $\beta$ is given by

$$\beta = \frac{\gamma k_B 1.945 c}{1 \text{ eV}} = 156.62 \text{ km/s}$$

For the cosmological interesting neutrino mass range (around 1 eV), is the non-relativistic assumption appropriate at the present time? What about at the equality redshift you derived in the previous problem?

for a mass around 1 eV we find that

$$\langle v \rangle \approx \beta$$

this means that our non-relativistic assumptions are correct. For the equality redshift we find

$$T_\nu(t_{rm}) = \frac{10^{10}}{\sqrt{t_{rm}}} \left( \frac{10^{3/4}}{g^*} \right)^{1/4} t_{rm} \approx 44,000 \text{ years} \quad g^* = \frac{21}{4}$$

so the temperature is

$$T_\nu = 8.65 \times 10^3 \text{K}$$

therefore the velocity is

$$\langle v \rangle = \beta \left( \frac{1 \text{ eV}}{m_\nu} \right) \left( \frac{T_\nu}{1.945} \right) = 156.62 \text{ km/s} \times 8.65 \times 10^3 = 1.35 \times 10^6 \text{ km/s} > c$$

so the non-relativistic assumption is not valid at the time matter and radiation decoupled. This would seem reasonably wrong because we derived the velocity equation assuming non-relativistic regime.

---

**Problem #21 [Sound Waves and Jeans Length]**

At typical sea-level conditions, the density of air is $1.23 \times 10^{-3} \text{ gcm}^{-3}$ and the speed of sound is $3.4 \times 10^4 \text{ cm sec}^{-1}$. Find (a) the jeans length and comment on how it compares with the thickness of the atmosphere and if you expect Jeans instability to occur; (b) the fractional change in frequency due to the self-gravity of the air, for a sound wave with wavelength 1 meter.

the Jeans length is given by

$$\lambda_J = \frac{2\pi}{k_J}$$
where \( k_J \) is the Jeans wave number which is given by

\[
k_J = \sqrt{\frac{4\pi G \rho_0}{v_s^2}}
\]

where \( v_s^2 \) is the characteristic sound speed, thus the Jeans length is

\[
\lambda_J = \sqrt{\frac{\pi v_s^2}{G \rho_0}} = 6.65 \times 10^9 \text{cm}
\]

This means that Jeans instability will not occur, due to the fact that the thickness of the atmosphere is \( \ll \) than the Jeans wavelength.

(b). to find the fractional change in frequency we must use

\[
\frac{\Delta \omega}{\omega} = \frac{\omega - \omega_J}{\omega} = \frac{v_s k - v_s \sqrt{(k^2 - k_J^2)}}{v_s k} = 1 - \frac{\sqrt{(k^2 - k_J^2)}}{k}
\]

which yields

\[
\frac{\Delta \omega}{\omega} = 1 - \frac{\lambda}{2\pi} \left( \frac{(2\pi)^2}{\lambda^2} - \frac{4\pi G \rho_0}{v_s^2} \right)^{1/2}
\]

\[
\frac{\Delta \omega}{\omega} = 1 - \sqrt{1 - \frac{G \rho_0 \lambda^2}{\pi v_s^2}} = 1 - \sqrt{1 - 2.26 \times 10^{-16}} \approx 0
\]

**Problem #22 [No More Jeans Swindle]**

The Jeans instability can be analyzed exactly, without invoking the Jeans swindle, in certain cylindrical rotating systems. Consider a homogeneous, self-gravitating fluid of density \( \rho_0 \), contained in an infinite cylinder of radius \( R_0 \). The cylinder walls and fluid rotate at uniform angular speed \( \vec{\Omega} = \Omega \hat{z} \), where \( \hat{z} \) lies along the axis of the cylinder. The Euler equation for this rotating system is

\[
\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = -\frac{1}{\rho} \vec{\nabla} P - \vec{\nabla} \phi - 2 \vec{\Omega} \times \vec{v} + \Omega^2 (x \hat{x} + y \hat{y})
\]

where the additional terms are the Coriolis and centrifugal forces.

(a). Show that the gravitational force per unit mass inside the cylinder is

\[
-\vec{\nabla} \phi_0 = -2\pi G \rho_0 (x \hat{x} + y \hat{y})
\]
We can solve this problem by using Gauss’s law, which states

\[ F_g \cdot A = 4\pi G M_{\text{enc}} \]

where the \( M_{\text{enc}} \) and the \( A \) are given by

\[ M_{\text{enc}} = \pi R_0^2 h \rho_0 \]
\[ A = 2\pi R_0 h \]

this gives us

\[ F_g(2\pi R_0 h) = 4\pi G (\pi R_0^2 h \rho_0) \]
\[ |F_g| = 2\pi G \rho_0 R_0 \]

but we know that

\[ \vec{F} = -|F|\hat{r} = -2\pi \rho_0 G R_0 \hat{r} \]

but \( R_0 \hat{r} = (x\hat{x} + y\hat{y}) \), so we find

\[ \vec{F} = -\nabla \phi_0 = -2\pi \rho_0 (x\hat{x} + y\hat{y}) \]

(b). Find the condition on \( \Omega \) so that the fluid is in equilibrium with zero velocity and no pressure gradients.

The conditions needed for this problem are

\[ \vec{v}_0 = 0 \quad \vec{v} P = 0 \]

The Euler equation is

\[ \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{1}{\rho} \vec{v} P - \nabla \phi - 2\vec{\Omega} \times \vec{v} + \Omega^2 (x\hat{x} + y\hat{y}) \]

applying these conditions we find

\[ \nabla \phi = \Omega^2 (x\hat{x} + y\hat{y}) = 2\pi G \rho_0 (x\hat{x} + y\hat{y}) \]

thus, we find

\[ \Omega = \sqrt{2\pi G \rho_0} \]

(c). Let \( R_0 \to \infty \) so that the boundary condition due to the wall can be neglected. Find the dispersion relation for waves propagating parallel to the rotation axis \( \vec{z} \). Discuss if these waves are stable.
we know that
\[ \rho_1 = \rho_1 e^{i(kz - \omega t)} \]
\[ \vec{v}_1 = (v_x \hat{x} + v_y \hat{y} + v_z \hat{z}) e^{i(kz - \omega t)} \]
\[ P_1 = v_x^2 \rho_1 \]
\[ \phi_1 = \nabla^2 \phi_1 = 4\pi G \rho_1 \]

We must use these relationships to linearize the three fluid equation, the linearized equation are given as

\[ \frac{\partial \vec{v}_1}{\partial t} = \frac{1}{\rho_0} \vec{\nabla} P_1 - \vec{\nabla} \phi_1 - 2\vec{\Omega} \times \vec{v}_1 \text{ equation 1} \]
\[ \frac{\partial \rho_1}{\partial t} = -\rho_0 (\vec{\nabla} \cdot \vec{v}_1) \text{ equation 2} \]
\[ \vec{\nabla}^2 \phi_1 = 4\pi G \rho_1 \text{ equation 3} \]

We can take a time derivative of equation 2 to get

\[ \frac{\partial^2 \rho_1}{\partial t^2} = -\rho_0 \frac{\partial (\vec{\nabla} \cdot \vec{v}_1)}{\partial t} = -\rho_0 \vec{\nabla} \cdot \left( \frac{1}{\rho_0} \vec{\nabla} P_1 - \vec{\nabla} \phi_1 - 2\vec{\Omega} \times \vec{v}_1 \right) \]

which gives us

\[ \frac{\partial^2 \rho_1}{\partial t^2} = -\rho_0 \left[ \frac{1}{\rho_0} \vec{\nabla}^2 P_1 - \vec{\nabla}^2 \phi_1 - 2\vec{\nabla} \cdot (\Omega v_x \hat{y} - \Omega v_y \hat{x}) e^{i(kz - \omega t)} \right] \]
\[ = -\rho_0 \left[ \frac{1}{\rho_0} \vec{\nabla}^2 P_1 - \vec{\nabla}^2 \phi_1 \right] \]
\[ = -\vec{\nabla}^2 P_1 + \rho_0 \vec{\nabla}^2 \phi_1 \]

since we know that

\[ P_1 = v_x^2 \rho_1 \quad \nabla^2 \phi_1 = 4\pi G \rho_1 \]

we can just plug this in to find

\[ \frac{\partial^2 \rho_1}{\partial t^2} = -v_x^2 \vec{\nabla}^2 \rho_1 + \rho_0 4\pi G \rho_1 \]

since we also know that

\[ \rho_1 = \rho_1 e^{i(kz - \omega t)} \]

we find

\[ \frac{\partial^2 \rho_1}{\partial t^2} = -\omega^2 \rho_1 e^{i(kz - \omega t)} = -\omega^2 \rho_1 \]
\[ \vec{\nabla}^2 \rho_1 = (-k)^2 \rho_1 e^{i(kz - \omega t)} = k^2 \rho_1 \]

therefore we find

\[ -\omega^2 \rho_1 = -v_x^2 k^2 \rho_1 + \rho_0 4\pi G \rho_1 \]
thus we find the dispersion relationship to be

\[ \omega^2 = v_x^2 k^2 - \rho_0 4\pi G \rho_1 \]

(d). Find the dispersion relation for waves propagating **perpendicular** (you may pick \( \vec{x} \) without loss of generality) to the rotation axis \( \vec{z} \). Discuss if these waves are stable.

We will solve this problem the same way as part (c), we can begin with

\[ \frac{\partial^2 \rho_1}{\partial t^2} = -\rho_0 \frac{\partial (\vec{v} \cdot \vec{v}_1)}{\partial t} = -\rho_0 \vec{v} \cdot \left( \frac{1}{\rho_0} \vec{v} P_1 - \vec{v} \phi_1 - 2\Omega \times \vec{v}_1 \right) \]

we need to solve for

\[ \vec{\Omega} \times \vec{v}_1 = \vec{v}_1 \cdot \nabla \times \vec{\Omega} - \vec{\Omega} \cdot \vec{\nabla} \times \vec{v}_1 = -\vec{\Omega} \cdot \vec{\nabla} \times \vec{v}_1 \]

so we find

\[ \frac{\partial^2 \rho_1}{\partial t^2} = -\rho_0 \frac{\partial (\vec{v} \cdot \vec{v}_1)}{\partial t} = -\rho_0 \vec{v} \cdot \left( \frac{1}{\rho_0} \vec{v} P_1 - \vec{v} \phi_1 - 2(-\vec{\Omega} \cdot \vec{\nabla} \times \vec{v}_1) \right) \]

but we know that

\[ \frac{\partial \vec{v}_1}{\partial t} = \frac{1}{\rho_0} \vec{v} P_1 - \vec{v} \phi_1 - 2\vec{\Omega} \times \vec{v}_1 \]

so

\[ \vec{v} \times \frac{\partial \vec{v}_1}{\partial t} = -2\vec{v} \times \vec{\Omega} \times \vec{v}_1 = -2\vec{v} \times (\Omega v_y \hat{y} - \Omega v_x \hat{x}) = -2\Omega \left( -\frac{\partial v_x}{\partial z} \hat{z} - \frac{\partial v_y}{\partial z} \hat{y} + \left( \frac{\partial v_x}{\partial x} - \frac{\partial v_y}{\partial y} \right) \hat{z} \right) \]

\[ = -2\Omega \frac{\partial v_x}{\partial x} \hat{z} = \frac{d}{dt} \vec{v} \times \vec{v}_1 \]

we also know that

\[ \frac{\partial \rho_1}{\partial t} = -\rho_0 \vec{v} \cdot \vec{v}_1 = -\rho_0 \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) \]

\[ = \rho_0 \frac{\partial v_x}{2\Omega} \frac{\partial v_x}{\partial x} \]

and so

\[ \rho_1 = \frac{\rho_0}{2\Omega} (\vec{v} \times \vec{v}_1) z \]

therefore

\[ \vec{\Omega} \cdot (\vec{v} \times \vec{v}_1) = \Omega (\nabla \times \vec{v}_1) z = \frac{2\Omega^2 \rho_1}{\rho_0} \]
thus
\[
\frac{\partial^2 \rho_1}{\partial t^2} = -\rho_0 \left[ \frac{1}{\rho_0} \nabla^2 P_1 - \nabla^2 \phi_1 - 2 \left( \frac{2\Omega^2 \rho_1}{\rho_0} \right) \right]
\]
and as before
\[
-\omega^2 \rho_1 = -\nabla^2 P_1 + \rho_0 \nabla^2 \phi_1 + 2\rho_0 \left( \frac{2\Omega^2 \rho_1}{\rho_0} \right)
-\omega^2 = -v_s^2 k^2 + \rho_0 4\pi G + 4\Omega^2
\]
Thus we find the dispersion relationship to be
\[
\omega^2 = v_s^2 k^2 - 4\pi G \rho_0 - 4\Omega^2
\]

Problem #23 [Structure Formation I: Radiation Era]

In class we derived the time evolution of density perturbations in non-relativistic matter \(\delta_m\) (above the Jeans length) in the matter dominated era. Here we will find the behavior of \(\delta_m\) in the radiation dominated era. Assume a flat, \(\Omega_\Lambda = 0\) universe for simplicity.

(a). Assume that the perturbation in radiation, \(\delta_r\), is negligible. Show that the linearized evolution equation for \(\delta_m\) can be written as
\[
\frac{d^2 \delta_m}{dy^2} + \frac{2 + 3y}{2y(1+y)} \frac{d\delta_m}{dy} - \frac{3}{2y(1+y)} \delta_m = 0
\]
where \(y \equiv \rho_m/\rho_r = a/a_{eq}\) is the equality time.

The linearized equation that we need to solve is given as
\[
\ddot{\delta}_m + \frac{2}{a} \dot{\delta}_m - 4\pi G \rho \delta_m = 0
\]
We know that
\[
\delta_m = \delta_m \quad \delta_m = \frac{d\delta_m}{dy} \frac{\dot{\delta}}{y} \quad \delta_m = \frac{d^2 \delta_m}{dy^2} (\dot{y})^2 + \frac{d\delta_m}{dy} \frac{\dot{\delta}}{y}
\]
we also know
\[
\frac{\dot{y}}{y} = \frac{\dot{a}}{a} \quad \rho = \rho_m + \rho_r = \rho_m \left( 1 + \frac{\rho_r}{\rho_m} \right) = \rho_m \left( 1 + \frac{1}{y} \right) \quad 4\pi G \rho = \frac{3}{2} \left( \frac{\dot{y}}{y} \right)^2 \frac{y}{y+1}
\]
the double derivative of the scale factor is given as
\[
\ddot{a} = \ddot{y} = -\frac{4\pi}{3} G \rho (1 + 3w)y = -\frac{4\pi}{3} G \rho \left( 1 + \frac{2}{y} \right) y = -\frac{y + 2}{2y(y + 1)} (\dot{y})^2
\]
putting all this together we find an equation of the form

\[
\frac{d^2 \delta_m}{dy^2} (\dot{y})^2 + \frac{d\delta_m}{dy} \ddot{y} + \frac{2}{y} \frac{d\delta_m}{dy} (\dot{y})^2 - 4\pi G \rho \delta_m = 0
\]

\[
\frac{d^2 \delta_m}{dy^2} - \frac{y+2}{2y(y+1)} \frac{d\delta_m}{dy} + \frac{2}{y} \frac{d\delta_m}{dy} - \frac{3}{2y(y+1)} \delta_m = 0
\]

which yields

\[
\frac{d^2 \delta_m}{dy^2} + \frac{2+3y}{2y(y+1)} \frac{d\delta_m}{dy} - \frac{3}{2y(y+1)} \delta_m = 0
\]

(b). At late time \((y \gg 1)\), compare your solutions for \(\delta_m\) with those derived in class. if \(y \gg 1\) the universe becomes matter dominated and the above expression becomes

\[
\frac{d^2 \delta_m}{dy^2} + \frac{3}{2y} \frac{d\delta_m}{dy} - \frac{3}{2y^2} \delta_m = 0
\]

if we assume that this has a power solution of the form

\[
\delta_m \propto y^\alpha \quad \frac{d\delta_m}{dy} = \alpha y^{\alpha-1} \quad \frac{d^2 \delta_m}{dy^2} = \alpha (\alpha - 1) y^{\alpha-2}
\]

we find that

\[
\alpha^2 + \frac{1}{2} \alpha - \frac{3}{2} = 0
\]

which can be factored as

\[
(\alpha - 1) \left( \alpha + \frac{3}{2} \right) = 0
\]

giving the two solutions as

\[
\alpha = 1 \quad \alpha = -3/2
\]

and so we can see that

\[
\delta^+_m \propto y \quad \text{and} \quad \delta^-_m \propto y^{-3/2}
\]

and we find a growing solution and a decaying solution. The two solutions from class are

\[
\delta^+_m \propto a \quad \delta^-_m \propto a^{-3/2}
\]

and so we know that when \(y \gg 1\) implies \(\rho_m \gg \rho_r\) and we recover the solutions from class.

(c). Verify that \(\delta_m \propto y + 2/3\) is a solution to the equation in part (a) in general. Can \(\delta_m\) grow much in the radiation dominated era?

we can verify this solution by doing

\[
\delta_m = y + \frac{2}{3} \quad \delta'_m = 1 \quad \delta''_m = 0
\]
and plugging this into the differential equation yields
\[ \frac{2 + 3y}{2y(y + 1)} - \frac{3}{2y(y + 1)} \left( y + \frac{2}{3} \right) = 0 \]
\[ 0 = 0 \]

thus, this is a solution to the differential equation.

We know that for radiation dominated era \( y \ll 1 \) and since the solution for \( \delta_m^+ \) is linear we can see it does not grow very much during the radiation dominated regime.

**Problem #24 [Structure Formation II: \( \Omega_m \neq 1 \) Models]**

In class we derived the growths of linear density perturbations in matter for cosmological models with different \( \Omega_m \) (assuming \( \Omega_\Lambda = 0 \)).

(a). Plot the growing solution \( \delta_m(a) \) on a log-log scale for \( a=0.001 \) to \( 1 \) for four cosmological models: \( \Omega_m = 0.01, 0.1, 0.3, \) and \( 1 \). Normalize all your curves to \( \delta_m(a = 0.001) \). Make sure to plot all curves on a single figure so you can compare them.

The solutions to the linear equations governing the growth of density perturbation for the different cosmological models are given as

\[ \delta_m^+ \propto a \quad \Omega_0,m = 1 \]
\[ \delta_m^+ \propto \frac{3 \sinh \theta (\sinh \theta - \theta)}{(\cosh \theta - 1)^2} - 2 \quad \Omega_0,m < 1 \]
\[ a(\theta) = \frac{\Omega_0}{2(1-\Omega_0)}(\cosh \theta - 1) \quad \Omega_0,m < 1 \]

From these solutions we can solve \( \theta \) in terms of \( a \) and plug into the growing solution, i.e

\[ \theta' = \cosh^{-1} \left[ \frac{a(\theta)2(1-\Omega_0)}{\Omega_0} + 1 \right] \]

thus \( \delta_m^+ \) is given by

\[ \delta_m^+ \propto \frac{3 \sinh \theta'(\sinh \theta' - \theta')}{(\cosh \theta' - 1)^2} - 2 \]

the plot is given as
(b). How does the growth of perturbations depend on $\Omega_{0,m}$?

We can see that the growth of the fluctuations depend very strongly with $\Omega_{0,m}$.

For $\Omega_{0,m} = 0.01$ We can see that the growth of structure decreases very rapidly as $z \rightarrow 0$, where the slope of the curve approaches 0 today. This seems very unlikely, due to the fact that we live in a universe filled with galaxies.

For $\Omega_{0,m} = 0.10$ we can see that increasing the mass density by one order of magnitude, causes the growth of structure to continue as $z \rightarrow 0$ and we can see that the more matter you put into the universe the longer the growth of the fluctuations (over densities).

For $\Omega_{0,m} = 0.30$ we can see that this is very similar to the last curve, except that we can see now that it is approaching $\delta_m^+ \propto a$.

For $\Omega_{0,m} = 1.00$ we can see that the growth of structure is a constant

If the four models you plotted all produce the same level of perturbations today (that matches the observed number density of galaxies, for example), how do you expect (qualitatively) the amount of structures to differ in the four models at redshift $z = 3$?

To try to understand this question we can make the same plot as before, except that we must normalize all the models to $a(\theta) = 1.00$
thus from this plot we can see that at $z = 3$ the perturbations must have grown a lot faster in the past for $\Omega_{0,m} < 1$, with increasing growth as $\Omega_{0,m} \to 0$. This must have happened in order for us to see the structures we see today.

**Problem #25 [Particle Horizon and the Horizon Problem]**

The particle horizon at cosmic time $t$ is the physical distance that light has traveled since $t = 0$; it defines the maximum distances for causal communication:

$$d_H(t) = a(t) \int_0^r \frac{dr'}{\sqrt{1 - kr'^2}} = a(t) \int_0^t \frac{cdt'}{a(t')}$$

(16)

The integral can be solved exactly in the matter-dominated era for $\Omega_\Lambda = 0$

$$d_H(z) = \begin{cases} \frac{c}{H_0 \sqrt{1 - \Omega_{0,m}}} (1 + z)^{-1} \cosh^{-1} \left[ 1 + \frac{2(1 - \Omega_{0,m})}{(1 + z)\Omega_{0,m}} \right] & \Omega_{0,m} < 1 \\ \frac{2c}{H_0} (1 + z)^{-3/2} & \Omega_{0,m} = 1 \\ \frac{c}{H_0 \sqrt{\Omega_{0,m} - 1}} (1 + z)^{-1} \cos^{-1} \left[ 1 - \frac{2(\Omega_{0,m} - 1)}{(1 + z)\Omega_{0,m}} \right] & \Omega_{0,m} > 1 \end{cases}$$

(a). Verify that eq. (2) indeed gives the expression above for $\Omega_{0,m} = 1$ model. (Extra credit: verify the other two models.)
We know that
\[ d_H(t) = a(t) \int_0^t \frac{cdt'}{a(t)'} \quad \text{where } a(t) \propto t^{2/3}, \quad a(t)' \propto (t^{2/3})' \]
and this equation becomes
\[ d_H(t) = ct^{2/3} \int_0^t (t')^{-3/2} dt' = 3ct^{2/3}t^{1/3} = 3ct \]
but we know that
\[ a \propto \frac{t^{2/3}}{t_0^{2/3}}, \quad t \propto t_0^{a^{3/2}} \quad t_0 = \frac{2}{3H_0} \quad a = \frac{1}{1+z} \]
putting this all together we find
\[ t \propto \frac{2}{3H_0} (1+z)^{-3/2} \]
thus we derive
\[ d_H(z) = \frac{2c}{H_0} (1+z)^{-3/2} \]

(b). What is the present horizon size (in Mpc) if \( \Omega_{0,m} = 0.3, 1, \) and 3?

For the present horizon size we must set \( z = 0 \) into the above expressions. For \( \Omega_m = 0.3 \) we use
\[ d_H(z = 0) = \frac{c}{H_0 \sqrt{1-0.3}} \cosh^{-1} \left[ 1 + \frac{2(1-0.3)}{0.3} \right] = 8.7 \text{ Gpc}^{-1} \]
For \( \Omega_m = 1 \) we use
\[ d_H(z = 0) = \frac{2c}{H_0} = 6.0 \text{ Gpc}^{-1} \]
For \( \Omega_m = 3 \) we use
\[ d_H(z = 0) = \frac{c}{H_0 \sqrt{3-1}} \cos^{-1} \left[ 1 - \frac{2(3-1)}{3} \right] = 4.0 \text{ Gpc}^{-1} \]

(c). Show that at high redshifts \( 1+z \gg \Omega_{0,m}^{-1} \), a good approximation for all three cases is given by
\[ d_H(z) \approx \frac{2c}{\sqrt{\Omega_{0,m} H_0}} (1+z)^{-3/2} \]
(you may find the identity \( \cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}) \) useful.)
For $\Omega_{0,m} < 1$ we will use

$$d_H(z) = \frac{c}{H_0 \sqrt{1 - \Omega_{0,m}}}(1+z)^{-1}\cosh^{-1}(1+\alpha) \quad \text{where} \quad \alpha = \frac{2(1 - \Omega_{0,m})}{(1 + z)\Omega_{0,m}}$$

using the above relationship for $\cosh^{-1}$ we find

$$\cosh^{-1}(1+\alpha) = \ln[1 + \alpha + \sqrt{(1+\alpha)^2 - 1}] = \ln[1 + \alpha + \sqrt{2\alpha}]$$

but we know that

$$\sqrt{2\alpha} > \alpha$$

and we can just write

$$\cosh^{-1}(1+\alpha) = \ln[1 + \sqrt{2\alpha}]$$

and we know that the Taylor expansion for

$$\ln(1+x) \approx x$$

so we now get

$$\cosh^{-1}(1+\alpha) \approx \sqrt{2\alpha}$$

so now the distance horizon becomes

$$d_H(z) \approx \frac{c}{H_0 \sqrt{1 - \Omega_{0,m}}}(1+z)^{-1}\sqrt{2\alpha} \approx \frac{2c}{\sqrt{\Omega_{0,m}H_0}}(1+z)^{-3/2}$$

For $\Omega_{0,m} > 1$ we will use

$$d_H(z) = \frac{c}{H_0 \sqrt{\Omega_{0,m} - 1}}(1+z)^{-1}\cos^{-1}(1+\alpha) \quad \text{where} \quad \alpha = -\frac{2(\Omega_{0,m} - 1)}{(1 + z)\Omega_{0,m}}$$

we know that the Taylor expansion for $\cos(\theta)$ is given by

$$\cos(\theta) = 1 - \frac{\theta^2}{2} + ... \quad \rightarrow \quad \theta \approx \cos^{-1}\left(1 - \frac{\theta^2}{2}\right)$$

letting

$$\alpha = -\frac{\theta^2}{2}$$

we find

$$\theta = \cos^{-1}(1+\alpha) \approx \sqrt{-2\alpha}$$

and since we know what $\alpha$ is we can just rewrite this as

$$d_H(z) = \frac{c}{H_0 \sqrt{\Omega_{0,m} - 1}}(1+z)^{-1}\sqrt{2\alpha}$$
which is what we had before, i.e

$$d_H(z) \approx \frac{2c}{\sqrt{\Omega_{0,m}H_0}}(1+z)^{-3/2}$$

For $\Omega_{0,m} = 1$ we get the same thing as before, which is

$$d_H(z) = \frac{2c}{\sqrt{\Omega_{0,m}H_0}}(1+z)^{-3/2} = \frac{2c}{H_0}(1+z)^{-3/2}$$

(d). Show that the *comoving* size of the particle horizon at the time of photon decoupling can be written as $\lambda_{\text{dec}} = \beta(\Omega_{0,m}h^2)^{-1/2}$, and compute the value of $\beta$ in Mpc.

We know that

$$\lambda_{\text{dec}} = \frac{d_H(z)}{a_{\text{dec}}} = \frac{2c}{\sqrt{\Omega_{0,m}H_0}}(1+z_{\text{dec}})^{-3/2}(1+z_{\text{dec}}) = \frac{2c}{\sqrt{\Omega_{0,m}H_0}\sqrt{1+z_{\text{dec}}}}$$

this can be written in as

$$\lambda_{\text{dec}} = \frac{2c \text{ sMpc}}{100 \text{ km/s}}(\Omega_{0,m}h^2)^{-1/2} = \beta(\Omega_{0,m}h^2)^{-1/2}$$

where $\beta = \frac{2c \text{ sMpc}}{100 \text{ km/s}}$.

we also know that

$$z_{\text{dec}} = 1089$$

thus we find that $\beta$ is

$$\beta \approx 180 \text{ Mpc}$$

(e). Estimate the angular size (in degrees) subtended by $\lambda_{\text{dec}}$ on the sky today. This is called the horizon problem. Why is this a problem?

The angular size subtended in the sky is given by

$$\theta = \frac{\lambda_{\text{dec}}}{d_H(z = 0)}$$

for $\Omega_{0,m} = 1$

so we find

$$\theta = (1+z_{\text{dec}})^{-1/2} = .030 \text{ radians} = 1.72 \text{ degrees}$$

This is a problem because it states that things are causally connected by 1.7 degrees in the sky, but yet we see that the CMB is very uniform at scales larger than this, which begs the question: How do particles (which are not causally connected) have such a uniform temperature? This is a problem that Inflation theory has been able to answer.